

---

# Contents

<i>Preface to the third edition</i>	page xx
<i>Preface to the second edition</i>	xxiii
<i>Preface to the first edition</i>	xxv
<b>1 Preliminary algebra</b>	<b>1</b>
1.1 Simple functions and equations	1
<i>Polynomial equations; factorisation; properties of roots</i>	
1.2 Trigonometric identities	10
<i>Single angle; compound angles; double- and half-angle identities</i>	
1.3 Coordinate geometry	15
1.4 Partial fractions	18
<i>Complications and special cases</i>	
1.5 Binomial expansion	25
1.6 Properties of binomial coefficients	27
1.7 Some particular methods of proof	30
<i>Proof by induction; proof by contradiction; necessary and sufficient conditions</i>	
1.8 Exercises	36
1.9 Hints and answers	39
<b>2 Preliminary calculus</b>	<b>41</b>
2.1 Differentiation	41
<i>Differentiation from first principles; products; the chain rule; quotients; implicit differentiation; logarithmic differentiation; Leibnitz' theorem; special points of a function; curvature; theorems of differentiation</i>	

## CONTENTS

2.2	Integration	59
	<i>Integration from first principles; the inverse of differentiation; by inspection; sinusoidal functions; logarithmic integration; using partial fractions; substitution method; integration by parts; reduction formulae; infinite and improper integrals; plane polar coordinates; integral inequalities; applications of integration</i>	
2.3	Exercises	76
2.4	Hints and answers	81
<b>3</b>	<b>Complex numbers and hyperbolic functions</b>	<b>83</b>
3.1	The need for complex numbers	83
3.2	Manipulation of complex numbers	85
	<i>Addition and subtraction; modulus and argument; multiplication; complex conjugate; division</i>	
3.3	Polar representation of complex numbers	92
	<i>Multiplication and division in polar form</i>	
3.4	de Moivre's theorem	95
	<i>trigonometric identities; finding the <math>n</math>th roots of unity; solving polynomial equations</i>	
3.5	Complex logarithms and complex powers	99
3.6	Applications to differentiation and integration	101
3.7	Hyperbolic functions	102
	<i>Definitions; hyperbolic-trigonometric analogies; identities of hyperbolic functions; solving hyperbolic equations; inverses of hyperbolic functions; calculus of hyperbolic functions</i>	
3.8	Exercises	109
3.9	Hints and answers	113
<b>4</b>	<b>Series and limits</b>	<b>115</b>
4.1	Series	115
4.2	Summation of series	116
	<i>Arithmetic series; geometric series; arithmetico-geometric series; the difference method; series involving natural numbers; transformation of series</i>	
4.3	Convergence of infinite series	124
	<i>Absolute and conditional convergence; series containing only real positive terms; alternating series test</i>	
4.4	Operations with series	131
4.5	Power series	131
	<i>Convergence of power series; operations with power series</i>	
4.6	Taylor series	136
	<i>Taylor's theorem; approximation errors; standard Maclaurin series</i>	
4.7	Evaluation of limits	141
4.8	Exercises	144
4.9	Hints and answers	149

## CONTENTS

<b>5</b>	<b>Partial differentiation</b>	<b>151</b>
5.1	Definition of the partial derivative	151
5.2	The total differential and total derivative	153
5.3	Exact and inexact differentials	155
5.4	Useful theorems of partial differentiation	157
5.5	The chain rule	157
5.6	Change of variables	158
5.7	Taylor's theorem for many-variable functions	160
5.8	Stationary values of many-variable functions	162
5.9	Stationary values under constraints	167
5.10	Envelopes	173
5.11	Thermodynamic relations	176
5.12	Differentiation of integrals	178
5.13	Exercises	179
5.14	Hints and answers	185
<b>6</b>	<b>Multiple integrals</b>	<b>187</b>
6.1	Double integrals	187
6.2	Triple integrals	190
6.3	Applications of multiple integrals	191
	<i>Areas and volumes; masses, centres of mass and centroids; Pappus' theorems; moments of inertia; mean values of functions</i>	
6.4	Change of variables in multiple integrals	199
	<i>Change of variables in double integrals; evaluation of the integral <math>I = \int_{-\infty}^{\infty} e^{-x^2} dx</math>; change of variables in triple integrals; general properties of Jacobians</i>	
6.5	Exercises	207
6.6	Hints and answers	211
<b>7</b>	<b>Vector algebra</b>	<b>212</b>
7.1	Scalars and vectors	212
7.2	Addition and subtraction of vectors	213
7.3	Multiplication by a scalar	214
7.4	Basis vectors and components	217
7.5	Magnitude of a vector	218
7.6	Multiplication of vectors	219
	<i>Scalar product; vector product; scalar triple product; vector triple product</i>	

## CONTENTS

7.7	Equations of lines, planes and spheres	226
7.8	Using vectors to find distances <i>Point to line; point to plane; line to line; line to plane</i>	229
7.9	Reciprocal vectors	233
7.10	Exercises	234
7.11	Hints and answers	240
<b>8</b>	<b>Matrices and vector spaces</b>	<b>241</b>
8.1	Vector spaces <i>Basis vectors; inner product; some useful inequalities</i>	242
8.2	Linear operators	247
8.3	Matrices	249
8.4	Basic matrix algebra <i>Matrix addition; multiplication by a scalar; matrix multiplication</i>	250
8.5	Functions of matrices	255
8.6	The transpose of a matrix	255
8.7	The complex and Hermitian conjugates of a matrix	256
8.8	The trace of a matrix	258
8.9	The determinant of a matrix <i>Properties of determinants</i>	259
8.10	The inverse of a matrix	263
8.11	The rank of a matrix	267
8.12	Special types of square matrix <i>Diagonal; triangular; symmetric and antisymmetric; orthogonal; Hermitian and anti-Hermitian; unitary; normal</i>	268
8.13	Eigenvectors and eigenvalues <i>Of a normal matrix; of Hermitian and anti-Hermitian matrices; of a unitary matrix; of a general square matrix</i>	272
8.14	Determination of eigenvalues and eigenvectors <i>Degenerate eigenvalues</i>	280
8.15	Change of basis and similarity transformations	282
8.16	Diagonalisation of matrices	285
8.17	Quadratic and Hermitian forms <i>Stationary properties of the eigenvectors; quadratic surfaces</i>	288
8.18	Simultaneous linear equations <i>Range; null space; <math>N</math> simultaneous linear equations in <math>N</math> unknowns; singular value decomposition</i>	292
8.19	Exercises	307
8.20	Hints and answers	314
<b>9</b>	<b>Normal modes</b>	<b>316</b>
9.1	Typical oscillatory systems	317
9.2	Symmetry and normal modes	322

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Table of Contents

[More information](#)

## CONTENTS

9.3	Rayleigh–Ritz method	327
9.4	Exercises	329
9.5	Hints and answers	332
<b>10</b>	<b>Vector calculus</b>	<b>334</b>
10.1	Differentiation of vectors <i>Composite vector expressions; differential of a vector</i>	334
10.2	Integration of vectors	339
10.3	Space curves	340
10.4	Vector functions of several arguments	344
10.5	Surfaces	345
10.6	Scalar and vector fields	347
10.7	Vector operators <i>Gradient of a scalar field; divergence of a vector field; curl of a vector field</i>	347
10.8	Vector operator formulae <i>Vector operators acting on sums and products; combinations of grad, div and curl</i>	354
10.9	Cylindrical and spherical polar coordinates	357
10.10	General curvilinear coordinates	364
10.11	Exercises	369
10.12	Hints and answers	375
<b>11</b>	<b>Line, surface and volume integrals</b>	<b>377</b>
11.1	Line integrals <i>Evaluating line integrals; physical examples; line integrals with respect to a scalar</i>	377
11.2	Connectivity of regions	383
11.3	Green's theorem in a plane	384
11.4	Conservative fields and potentials	387
11.5	Surface integrals <i>Evaluating surface integrals; vector areas of surfaces; physical examples</i>	389
11.6	Volume integrals <i>Volumes of three-dimensional regions</i>	396
11.7	Integral forms for grad, div and curl	398
11.8	Divergence theorem and related theorems <i>Green's theorems; other related integral theorems; physical applications</i>	401
11.9	Stokes' theorem and related theorems <i>Related integral theorems; physical applications</i>	406
11.10	Exercises	409
11.11	Hints and answers	414
<b>12</b>	<b>Fourier series</b>	<b>415</b>
12.1	The Dirichlet conditions	415

## CONTENTS

12.2	The Fourier coefficients	417
12.3	Symmetry considerations	419
12.4	Discontinuous functions	420
12.5	Non-periodic functions	422
12.6	Integration and differentiation	424
12.7	Complex Fourier series	424
12.8	Parseval's theorem	426
12.9	Exercises	427
12.10	Hints and answers	431
<b>13</b>	<b>Integral transforms</b>	<b>433</b>
13.1	Fourier transforms <i>The uncertainty principle; Fraunhofer diffraction; the Dirac <math>\delta</math>-function; relation of the <math>\delta</math>-function to Fourier transforms; properties of Fourier transforms; odd and even functions; convolution and deconvolution; correlation functions and energy spectra; Parseval's theorem; Fourier transforms in higher dimensions</i>	433
13.2	Laplace transforms <i>Laplace transforms of derivatives and integrals; other properties of Laplace transforms</i>	453
13.3	Concluding remarks	459
13.4	Exercises	460
13.5	Hints and answers	466
<b>14</b>	<b>First-order ordinary differential equations</b>	<b>468</b>
14.1	General form of solution	469
14.2	First-degree first-order equations <i>Separable-variable equations; exact equations; inexact equations, integrating factors; linear equations; homogeneous equations; isobaric equations; Bernoulli's equation; miscellaneous equations</i>	470
14.3	Higher-degree first-order equations <i>Equations soluble for <math>p</math>; for <math>x</math>; for <math>y</math>; Clairaut's equation</i>	480
14.4	Exercises	484
14.5	Hints and answers	488
<b>15</b>	<b>Higher-order ordinary differential equations</b>	<b>490</b>
15.1	Linear equations with constant coefficients <i>Finding the complementary function <math>y_c(x)</math>; finding the particular integral <math>y_p(x)</math>; constructing the general solution <math>y_c(x) + y_p(x)</math>; linear recurrence relations; Laplace transform method</i>	492
15.2	Linear equations with variable coefficients <i>The Legendre and Euler linear equations; exact equations; partially known complementary function; variation of parameters; Green's functions; canonical form for second-order equations</i>	503

## CONTENTS

15.3	General ordinary differential equations <i>Dependent variable absent; independent variable absent; non-linear exact equations; isobaric or homogeneous equations; equations homogeneous in <math>x</math> or <math>y</math> alone; equations having <math>y = Ae^x</math> as a solution</i>	518
15.4	Exercises	523
15.5	Hints and answers	529
<b>16</b>	<b>Series solutions of ordinary differential equations</b>	<b>531</b>
16.1	Second-order linear ordinary differential equations <i>Ordinary and singular points</i>	531
16.2	Series solutions about an ordinary point	535
16.3	Series solutions about a regular singular point <i>Distinct roots not differing by an integer; repeated root of the indicial equation; distinct roots differing by an integer</i>	538
16.4	Obtaining a second solution <i>The Wronskian method; the derivative method; series form of the second solution</i>	544
16.5	Polynomial solutions	548
16.6	Exercises	550
16.7	Hints and answers	553
<b>17</b>	<b>Eigenfunction methods for differential equations</b>	<b>554</b>
17.1	Sets of functions <i>Some useful inequalities</i>	556
17.2	Adjoint, self-adjoint and Hermitian operators	559
17.3	Properties of Hermitian operators <i>Reality of the eigenvalues; orthogonality of the eigenfunctions; construction of real eigenfunctions</i>	561
17.4	Sturm–Liouville equations <i>Valid boundary conditions; putting an equation into Sturm–Liouville form</i>	564
17.5	Superposition of eigenfunctions: Green’s functions	569
17.6	A useful generalisation	572
17.7	Exercises	573
17.8	Hints and answers	576
<b>18</b>	<b>Special functions</b>	<b>577</b>
18.1	Legendre functions <i>General solution for integer <math>\ell</math>; properties of Legendre polynomials</i>	577
18.2	Associated Legendre functions	587
18.3	Spherical harmonics	593
18.4	Chebyshev functions	595
18.5	Bessel functions <i>General solution for non-integer <math>\nu</math>; general solution for integer <math>\nu</math>; properties of Bessel functions</i>	602

## CONTENTS

18.6	Spherical Bessel functions	614
18.7	Laguerre functions	616
18.8	Associated Laguerre functions	621
18.9	Hermite functions	624
18.10	Hypergeometric functions	628
18.11	Confluent hypergeometric functions	633
18.12	The gamma function and related functions	635
18.13	Exercises	640
18.14	Hints and answers	646
<b>19</b>	<b>Quantum operators</b>	<b>648</b>
19.1	Operator formalism <i>Commutators</i>	648
19.2	Physical examples of operators <i>Uncertainty principle; angular momentum; creation and annihilation operators</i>	656
19.3	Exercises	671
19.4	Hints and answers	674
<b>20</b>	<b>Partial differential equations: general and particular solutions</b>	<b>675</b>
20.1	Important partial differential equations <i>The wave equation; the diffusion equation; Laplace's equation; Poisson's equation; Schrödinger's equation</i>	676
20.2	General form of solution	680
20.3	General and particular solutions <i>First-order equations; inhomogeneous equations and problems; second-order equations</i>	681
20.4	The wave equation	693
20.5	The diffusion equation	695
20.6	Characteristics and the existence of solutions <i>First-order equations; second-order equations</i>	699
20.7	Uniqueness of solutions	705
20.8	Exercises	707
20.9	Hints and answers	711
<b>21</b>	<b>Partial differential equations: separation of variables and other methods</b>	<b>713</b>
21.1	Separation of variables: the general method	713
21.2	Superposition of separated solutions	717
21.3	Separation of variables in polar coordinates <i>Laplace's equation in polar coordinates; spherical harmonics; other equations in polar coordinates; solution by expansion; separation of variables for inhomogeneous equations</i>	725
21.4	Integral transform methods	747

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Table of Contents

[More information](#)

## CONTENTS

21.5	Inhomogeneous problems – Green’s functions <i>Similarities to Green’s functions for ordinary differential equations; general boundary-value problems; Dirichlet problems; Neumann problems</i>	751
21.6	Exercises	767
21.7	Hints and answers	773
<b>22</b>	<b>Calculus of variations</b>	<b>775</b>
22.1	The Euler–Lagrange equation	776
22.2	Special cases <i>F does not contain y explicitly; F does not contain x explicitly</i>	777
22.3	Some extensions <i>Several dependent variables; several independent variables; higher-order derivatives; variable end-points</i>	781
22.4	Constrained variation	785
22.5	Physical variational principles <i>Fermat’s principle in optics; Hamilton’s principle in mechanics</i>	787
22.6	General eigenvalue problems	790
22.7	Estimation of eigenvalues and eigenfunctions	792
22.8	Adjustment of parameters	795
22.9	Exercises	797
22.10	Hints and answers	801
<b>23</b>	<b>Integral equations</b>	<b>803</b>
23.1	Obtaining an integral equation from a differential equation	803
23.2	Types of integral equation	804
23.3	Operator notation and the existence of solutions	805
23.4	Closed-form solutions <i>Separable kernels; integral transform methods; differentiation</i>	806
23.5	Neumann series	813
23.6	Fredholm theory	815
23.7	Schmidt–Hilbert theory	816
23.8	Exercises	819
23.9	Hints and answers	823
<b>24</b>	<b>Complex variables</b>	<b>824</b>
24.1	Functions of a complex variable	825
24.2	The Cauchy–Riemann relations	827
24.3	Power series in a complex variable	830
24.4	Some elementary functions	832
24.5	Multivalued functions and branch cuts	835
24.6	Singularities and zeros of complex functions	837
24.7	Conformal transformations	839
24.8	Complex integrals	845

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Table of Contents

[More information](#)

## CONTENTS

24.9	Cauchy's theorem	849
24.10	Cauchy's integral formula	851
24.11	Taylor and Laurent series	853
24.12	Residue theorem	858
24.13	Definite integrals using contour integration	861
24.14	Exercises	867
24.15	Hints and answers	870
<b>25</b>	<b>Applications of complex variables</b>	<b>871</b>
25.1	Complex potentials	871
25.2	Applications of conformal transformations	876
25.3	Location of zeros	879
25.4	Summation of series	882
25.5	Inverse Laplace transform	884
25.6	Stokes' equation and Airy integrals	888
25.7	WKB methods	895
25.8	Approximations to integrals <i>Level lines and saddle points; steepest descents; stationary phase</i>	905
25.9	Exercises	920
25.10	Hints and answers	925
<b>26</b>	<b>Tensors</b>	<b>927</b>
26.1	Some notation	928
26.2	Change of basis	929
26.3	Cartesian tensors	930
26.4	First- and zero-order Cartesian tensors	932
26.5	Second- and higher-order Cartesian tensors	935
26.6	The algebra of tensors	938
26.7	The quotient law	939
26.8	The tensors $\delta_{ij}$ and $\epsilon_{ijk}$	941
26.9	Isotropic tensors	944
26.10	Improper rotations and pseudotensors	946
26.11	Dual tensors	949
26.12	Physical applications of tensors	950
26.13	Integral theorems for tensors	954
26.14	Non-Cartesian coordinates	955
26.15	The metric tensor	957
26.16	General coordinate transformations and tensors	960
26.17	Relative tensors	963
26.18	Derivatives of basis vectors and Christoffel symbols	965
26.19	Covariant differentiation	968
26.20	Vector operators in tensor form	971

---

 CONTENTS
 

---

26.21	Absolute derivatives along curves	975
26.22	Geodesics	976
26.23	Exercises	977
26.24	Hints and answers	982
<b>27</b>	<b>Numerical methods</b>	<b>984</b>
27.1	Algebraic and transcendental equations <i>Rearrangement of the equation; linear interpolation; binary chopping; Newton–Raphson method</i>	985
27.2	Convergence of iteration schemes	992
27.3	Simultaneous linear equations <i>Gaussian elimination; Gauss–Seidel iteration; tridiagonal matrices</i>	994
27.4	Numerical integration <i>Trapezium rule; Simpson’s rule; Gaussian integration; Monte Carlo methods</i>	1000
27.5	Finite differences	1019
27.6	Differential equations <i>Difference equations; Taylor series solutions; prediction and correction; Runge–Kutta methods; isoclines</i>	1020
27.7	Higher-order equations	1028
27.8	Partial differential equations	1030
27.9	Exercises	1033
27.10	Hints and answers	1039
<b>28</b>	<b>Group theory</b>	<b>1041</b>
28.1	Groups <i>Definition of a group; examples of groups</i>	1041
28.2	Finite groups	1049
28.3	Non-Abelian groups	1052
28.4	Permutation groups	1056
28.5	Mappings between groups	1059
28.6	Subgroups	1061
28.7	Subdividing a group <i>Equivalence relations and classes; congruence and cosets; conjugates and classes</i>	1063
28.8	Exercises	1070
28.9	Hints and answers	1074
<b>29</b>	<b>Representation theory</b>	<b>1076</b>
29.1	Dipole moments of molecules	1077
29.2	Choosing an appropriate formalism	1078
29.3	Equivalent representations	1084
29.4	Reducibility of a representation	1086
29.5	The orthogonality theorem for irreducible representations	1090

## CONTENTS

29.6	Characters	1092
	<i>Orthogonality property of characters</i>	
29.7	Counting irreps using characters	1095
	<i>Summation rules for irreps</i>	
29.8	Construction of a character table	1100
29.9	Group nomenclature	1102
29.10	Product representations	1103
29.11	Physical applications of group theory	1105
	<i>Bonding in molecules; matrix elements in quantum mechanics; degeneracy of normal modes; breaking of degeneracies</i>	
29.12	Exercises	1113
29.13	Hints and answers	1117
<b>30</b>	<b>Probability</b>	<b>1119</b>
30.1	Venn diagrams	1119
30.2	Probability	1124
	<i>Axioms and theorems; conditional probability; Bayes' theorem</i>	
30.3	Permutations and combinations	1133
30.4	Random variables and distributions	1139
	<i>Discrete random variables; continuous random variables</i>	
30.5	Properties of distributions	1143
	<i>Mean; mode and median; variance and standard deviation; moments; central moments</i>	
30.6	Functions of random variables	1150
30.7	Generating functions	1157
	<i>Probability generating functions; moment generating functions; characteristic functions; cumulant generating functions</i>	
30.8	Important discrete distributions	1168
	<i>Binomial; geometric; negative binomial; hypergeometric; Poisson</i>	
30.9	Important continuous distributions	1179
	<i>Gaussian; log-normal; exponential; gamma; chi-squared; Cauchy; Breit–Wigner; uniform</i>	
30.10	The central limit theorem	1195
30.11	Joint distributions	1196
	<i>Discrete bivariate; continuous bivariate; marginal and conditional distributions</i>	
30.12	Properties of joint distributions	1199
	<i>Means; variances; covariance and correlation</i>	
30.13	Generating functions for joint distributions	1205
30.14	Transformation of variables in joint distributions	1206
30.15	Important joint distributions	1207
	<i>Multinomial; multivariate Gaussian</i>	
30.16	Exercises	1211
30.17	Hints and answers	1219

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Table of Contents

[More information](#)

## CONTENTS

<b>31</b>	<b>Statistics</b>	<b>1221</b>
31.1	Experiments, samples and populations	1221
31.2	Sample statistics	1222
	<i>Averages; variance and standard deviation; moments; covariance and correlation</i>	
31.3	Estimators and sampling distributions	1229
	<i>Consistency, bias and efficiency; Fisher's inequality; standard errors; confidence limits</i>	
31.4	Some basic estimators	1243
	<i>Mean; variance; standard deviation; moments; covariance and correlation</i>	
31.5	Maximum-likelihood method	1255
	<i>ML estimator; transformation invariance and bias; efficiency; errors and confidence limits; Bayesian interpretation; large-N behaviour; extended ML method</i>	
31.6	The method of least squares	1271
	<i>Linear least squares; non-linear least squares</i>	
31.7	Hypothesis testing	1277
	<i>Simple and composite hypotheses; statistical tests; Neyman–Pearson; generalised likelihood-ratio; Student's <math>t</math>; Fisher's <math>F</math>; goodness of fit</i>	
31.8	Exercises	1298
31.9	Hints and answers	1303
	<i>Index</i>	1305

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Table of Contents

[More information](#)

## CONTENTS

**I am the very Model for a Student Mathematical**

I am the very model for a student mathematical;  
 I've information rational, and logical and practical.  
 I know the laws of algebra, and find them quite symmetrical,  
 And even know the meaning of 'a variate antithetical'.

I'm extremely well acquainted, with all things mathematical.  
 I understand equations, both the simple and quadratical.  
 About binomial theorems I'm teeming with a lot o'news,  
 With many cheerful facts about the square of the hypotenuse.

I'm very good at integral and differential calculus,  
 And solving paradoxes that so often seem to rankle us.  
 In short in matters rational, and logical and practical,  
 I am the very model for a student mathematical.

I know the singularities of equations differential,  
 And some of these are regular, but the rest are quite essential.  
 I quote the results of giants; with Euler, Newton, Gauss, Laplace,  
 And can calculate an orbit, given a centre, force and mass.

I can reconstruct equations, both canonical and formal,  
 And write all kinds of matrices, orthogonal, real and normal.  
 I show how to tackle problems that one has never met before,  
 By analogy or example, or with some clever metaphor.

I seldom use equivalence to help decide upon a class,  
 But often find an integral, using a contour o'er a pass.  
 In short in matters rational, and logical and practical,  
 I am the very model for a student mathematical.

---

When you have learnt just what is meant by 'Jacobian' and 'Abelian';  
 When you at sight can estimate, for the modal, mean and median;  
 When describing normal subgroups is much more than recitation;  
 When you understand precisely what is 'quantum excitation';

When you know enough statistics that you can recognise RV;  
 When you have learnt all advances that have been made in SVD;  
 And when you can spot the transform that solves some tricky PDE,  
 You will feel no better student has ever sat for a degree.

Your accumulated knowledge, whilst extensive and exemplary,  
 Will have only been brought down to the beginning of last century,  
 But still in matters rational, and logical and practical,  
 You'll be the very model of a student mathematical.

KFR, with apologies to W.S. Gilbert

## 1

---

## *Preliminary algebra*

This opening chapter reviews the basic algebra of which a working knowledge is presumed in the rest of the book. Many students will be familiar with much, if not all, of it, but recent changes in what is studied during secondary education mean that it cannot be taken for granted that they will already have a mastery of all the topics presented here. The reader may assess which areas need further study or revision by attempting the exercises at the end of the chapter. The main areas covered are polynomial equations and the related topic of partial fractions, curve sketching, coordinate geometry, trigonometric identities and the notions of proof by induction or contradiction.

### **1.1 Simple functions and equations**

It is normal practice when starting the mathematical investigation of a physical problem to assign an algebraic symbol to the quantity whose value is sought, either numerically or as an explicit algebraic expression. For the sake of definiteness, in this chapter we will use  $x$  to denote this quantity most of the time. Subsequent steps in the analysis involve applying a combination of known laws, consistency conditions and (possibly) given constraints to derive one or more equations satisfied by  $x$ . These equations may take many forms, ranging from a simple polynomial equation to, say, a partial differential equation with several boundary conditions. Some of the more complicated possibilities are treated in the later chapters of this book, but for the present we will be concerned with techniques for the solution of relatively straightforward algebraic equations.

#### ***1.1.1 Polynomials and polynomial equations***

Firstly we consider the simplest type of equation, a *polynomial equation*, in which a *polynomial* expression in  $x$ , denoted by  $f(x)$ , is set equal to zero and thereby

## PRELIMINARY ALGEBRA

forms an equation which is satisfied by particular values of  $x$ , called the *roots* of the equation:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0. \quad (1.1)$$

Here  $n$  is an integer  $> 0$ , called the *degree* of both the polynomial and the equation, and the known coefficients  $a_0, a_1, \dots, a_n$  are real quantities with  $a_n \neq 0$ .

Equations such as (1.1) arise frequently in physical problems, the coefficients  $a_i$  being determined by the physical properties of the system under study. What is needed is to find some or all of the roots of (1.1), i.e. the  $x$ -values,  $\alpha_k$ , that satisfy  $f(\alpha_k) = 0$ ; here  $k$  is an index that, as we shall see later, can take up to  $n$  different values, i.e.  $k = 1, 2, \dots, n$ . The roots of the polynomial equation can equally well be described as the zeros of the polynomial. When they are *real*, they correspond to the points at which a graph of  $f(x)$  crosses the  $x$ -axis. Roots that are complex (see chapter 3) do not have such a graphical interpretation.

For polynomial equations containing powers of  $x$  greater than  $x^4$  general methods do not exist for obtaining explicit expressions for the roots  $\alpha_k$ . Even for  $n = 3$  and  $n = 4$  the prescriptions for obtaining the roots are sufficiently complicated that it is usually preferable to obtain exact or approximate values by other methods. Only for  $n = 1$  and  $n = 2$  can closed-form solutions be given. These results will be well known to the reader, but they are given here for the sake of completeness. For  $n = 1$ , (1.1) reduces to the *linear* equation

$$a_1 x + a_0 = 0; \quad (1.2)$$

the solution (root) is  $\alpha_1 = -a_0/a_1$ . For  $n = 2$ , (1.1) reduces to the *quadratic* equation

$$a_2 x^2 + a_1 x + a_0 = 0; \quad (1.3)$$

the two roots  $\alpha_1$  and  $\alpha_2$  are given by

$$\alpha_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}. \quad (1.4)$$

When discussing specifically quadratic equations, as opposed to more general polynomial equations, it is usual to write the equation in one of the two notations

$$ax^2 + bx + c = 0, \quad ax^2 + 2bx + c = 0, \quad (1.5)$$

with respective explicit pairs of solutions

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - ac}}{a}. \quad (1.6)$$

Of course, these two notations are entirely equivalent and the only important point is to associate each form of answer with the corresponding form of equation; most people keep to one form, to avoid any possible confusion.

## 1.1 SIMPLE FUNCTIONS AND EQUATIONS

If the value of the quantity appearing under the square root sign is positive then both roots are real; if it is negative then the roots form a complex conjugate pair, i.e. they are of the form  $p \pm iq$  with  $p$  and  $q$  real (see chapter 3); if it has zero value then the two roots are equal and special considerations usually arise.

Thus linear and quadratic equations can be dealt with in a cut-and-dried way. We now turn to methods for obtaining partial information about the roots of higher-degree polynomial equations. In some circumstances the knowledge that an equation has a root lying in a certain range, or that it has no real roots at all, is all that is actually required. For example, in the design of electronic circuits it is necessary to know whether the current in a proposed circuit will break into spontaneous oscillation. To test this, it is sufficient to establish whether a certain polynomial equation, whose coefficients are determined by the physical parameters of the circuit, has a root with a positive real part (see chapter 3); complete determination of all the roots is not needed for this purpose. If the complete set of roots of a polynomial equation is required, it can usually be obtained to any desired accuracy by numerical methods such as those described in chapter 27.

There is no explicit step-by-step approach to finding the roots of a general polynomial equation such as (1.1). In most cases analytic methods yield only information *about* the roots, rather than their exact values. To explain the relevant techniques we will consider a particular example, ‘thinking aloud’ on paper and expanding on special points about methods and lines of reasoning. In more routine situations such comment would be absent and the whole process briefer and more tightly focussed.

*Example: the cubic case*

Let us investigate the roots of the equation

$$g(x) = 4x^3 + 3x^2 - 6x - 1 = 0 \quad (1.7)$$

or, in an alternative phrasing, investigate the zeros of  $g(x)$ . We note first of all that this is a *cubic* equation. It can be seen that for  $x$  large and positive  $g(x)$  will be large and positive and, equally, that for  $x$  large and negative  $g(x)$  will be large and negative. Therefore, intuitively (or, more formally, by continuity)  $g(x)$  must cross the  $x$ -axis at least once and so  $g(x) = 0$  must have at least one real root. Furthermore, it can be shown that if  $f(x)$  is an  $n$ th-degree polynomial then the graph of  $f(x)$  must cross the  $x$ -axis an even or odd number of times as  $x$  varies between  $-\infty$  and  $+\infty$ , according to whether  $n$  itself is even or odd. Thus a polynomial of odd degree always has at least one real root, but one of even degree may have no real root. A small complication, discussed later in this section, occurs when repeated roots arise.

Having established that  $g(x) = 0$  has at least one real root, we may ask how

## PRELIMINARY ALGEBRA

many real roots it *could* have. To answer this we need one of the fundamental theorems of algebra, mentioned above:

An  $n$ th-degree polynomial equation has exactly  $n$  roots.

It should be noted that this does not imply that there are  $n$  *real* roots (only that there are not more than  $n$ ); some of the roots may be of the form  $p + iq$ .

To make the above theorem plausible and to see what is meant by repeated roots, let us suppose that the  $n$ th-degree polynomial equation  $f(x) = 0$ , (1.1), has  $r$  roots  $\alpha_1, \alpha_2, \dots, \alpha_r$ , considered distinct for the moment. That is, we suppose that  $f(\alpha_k) = 0$  for  $k = 1, 2, \dots, r$ , so that  $f(x)$  vanishes only when  $x$  is equal to one of the  $r$  values  $\alpha_k$ . But the same can be said for the function

$$F(x) = A(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_r), \quad (1.8)$$

in which  $A$  is a non-zero constant;  $F(x)$  can clearly be multiplied out to form a polynomial expression.

We now call upon a second fundamental result in algebra: that if two polynomial functions  $f(x)$  and  $F(x)$  have equal values for *all* values of  $x$ , then their coefficients are equal on a term-by-term basis. In other words, we can equate the coefficients of each and every power of  $x$  in the two expressions (1.8) and (1.1); in particular we can equate the coefficients of the highest power of  $x$ . From this we have  $Ax^r \equiv a_n x^n$  and thus that  $r = n$  and  $A = a_n$ . As  $r$  is both equal to  $n$  and to the number of roots of  $f(x) = 0$ , we conclude that the  $n$ th-degree polynomial  $f(x) = 0$  has  $n$  roots. (Although this line of reasoning may make the theorem plausible, it does not constitute a proof since we have not shown that it is permissible to write  $f(x)$  in the form of equation (1.8).)

We next note that the condition  $f(\alpha_k) = 0$  for  $k = 1, 2, \dots, r$ , could also be met if (1.8) were replaced by

$$F(x) = A(x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \cdots (x - \alpha_r)^{m_r}, \quad (1.9)$$

with  $A = a_n$ . In (1.9) the  $m_k$  are integers  $\geq 1$  and are known as the multiplicities of the roots,  $m_k$  being the multiplicity of  $\alpha_k$ . Expanding the right-hand side (RHS) leads to a polynomial of degree  $m_1 + m_2 + \cdots + m_r$ . This sum must be equal to  $n$ . Thus, if any of the  $m_k$  is greater than unity then the number of *distinct* roots,  $r$ , is less than  $n$ ; the total number of roots remains at  $n$ , but one or more of the  $\alpha_k$  counts more than once. For example, the equation

$$F(x) = A(x - \alpha_1)^2(x - \alpha_2)^3(x - \alpha_3)(x - \alpha_4) = 0$$

has exactly seven roots,  $\alpha_1$  being a double root and  $\alpha_2$  a triple root, whilst  $\alpha_3$  and  $\alpha_4$  are unrepeated (*simple*) roots.

We can now say that our particular equation (1.7) has either one or three real roots but in the latter case it may be that not all the roots are distinct. To decide how many real roots the equation has, we need to anticipate two ideas from the

## 1.1 SIMPLE FUNCTIONS AND EQUATIONS

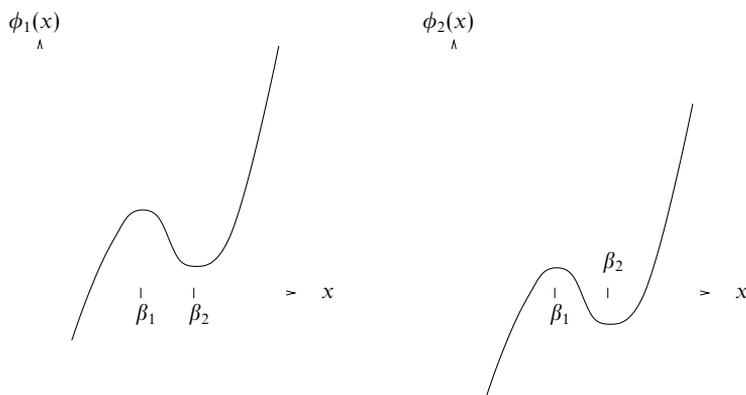


Figure 1.1 Two curves  $\phi_1(x)$  and  $\phi_2(x)$ , both with zero derivatives at the same values of  $x$ , but with different numbers of real solutions to  $\phi_i(x) = 0$ .

next chapter. The first of these is the notion of the derivative of a function, and the second is a result known as Rolle's theorem.

The *derivative*  $f'(x)$  of a function  $f(x)$  measures the slope of the tangent to the graph of  $f(x)$  at that value of  $x$  (see figure 2.1 in the next chapter). For the moment, the reader with no prior knowledge of calculus is asked to accept that the derivative of  $ax^n$  is  $nax^{n-1}$ , so that the derivative  $g'(x)$  of the curve  $g(x) = 4x^3 + 3x^2 - 6x - 1$  is given by  $g'(x) = 12x^2 + 6x - 6$ . Similar expressions for the derivatives of other polynomials are used later in this chapter.

Rolle's theorem states that if  $f(x)$  has equal values at two different values of  $x$  then at some point between these two  $x$ -values its derivative is equal to zero; i.e. the tangent to its graph is parallel to the  $x$ -axis at that point (see figure 2.2).

Having briefly mentioned the derivative of a function and Rolle's theorem, we now use them to establish whether  $g(x)$  has one or three real zeros. If  $g(x) = 0$  does have three real roots  $\alpha_k$ , i.e.  $g(\alpha_k) = 0$  for  $k = 1, 2, 3$ , then it follows from Rolle's theorem that between any consecutive pair of them (say  $\alpha_1$  and  $\alpha_2$ ) there must be some real value of  $x$  at which  $g'(x) = 0$ . Similarly, there must be a further zero of  $g'(x)$  lying between  $\alpha_2$  and  $\alpha_3$ . Thus a *necessary* condition for three real roots of  $g(x) = 0$  is that  $g'(x) = 0$  itself has two real roots.

However, this condition on the number of roots of  $g'(x) = 0$ , whilst necessary, is not *sufficient* to guarantee three real roots of  $g(x) = 0$ . This can be seen by inspecting the cubic curves in figure 1.1. For each of the two functions  $\phi_1(x)$  and  $\phi_2(x)$ , the derivative is equal to zero at both  $x = \beta_1$  and  $x = \beta_2$ . Clearly, though,  $\phi_2(x) = 0$  has three real roots whilst  $\phi_1(x) = 0$  has only one. It is easy to see that the crucial difference is that  $\phi_1(\beta_1)$  and  $\phi_1(\beta_2)$  have the same sign, whilst  $\phi_2(\beta_1)$  and  $\phi_2(\beta_2)$  have opposite signs.

It will be apparent that for some equations,  $\phi(x) = 0$  say,  $\phi'(x)$  equals zero

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Excerpt

[More information](#)

## PRELIMINARY ALGEBRA

at a value of  $x$  for which  $\phi(x)$  is also zero. Then the graph of  $\phi(x)$  just touches the  $x$ -axis. When this happens the value of  $x$  so found is, in fact, a double real root of the polynomial equation (corresponding to one of the  $m_k$  in (1.9) having the value 2) and must be counted twice when determining the number of real roots.

Finally, then, we are in a position to decide the number of real roots of the equation

$$g(x) = 4x^3 + 3x^2 - 6x - 1 = 0.$$

The equation  $g'(x) = 0$ , with  $g'(x) = 12x^2 + 6x - 6$ , is a quadratic equation with explicit solutions<sup>§</sup>

$$\beta_{1,2} = \frac{-3 \pm \sqrt{9 + 72}}{12},$$

so that  $\beta_1 = -1$  and  $\beta_2 = \frac{1}{2}$ . The corresponding values of  $g(x)$  are  $g(\beta_1) = 4$  and  $g(\beta_2) = -\frac{11}{4}$ , which are of opposite sign. This indicates that  $4x^3 + 3x^2 - 6x - 1 = 0$  has three real roots, one lying in the range  $-1 < x < \frac{1}{2}$  and the others one on each side of that range.

The techniques we have developed above have been used to tackle a cubic equation, but they can be applied to polynomial equations  $f(x) = 0$  of degree greater than 3. However, much of the analysis centres around the equation  $f'(x) = 0$  and this itself, being then a polynomial equation of degree 3 or more, either has no closed-form general solution or one that is complicated to evaluate. Thus the amount of information that can be obtained about the roots of  $f(x) = 0$  is correspondingly reduced.

*A more general case*

To illustrate what can (and cannot) be done in the more general case we now investigate as far as possible the real roots of

$$f(x) = x^7 + 5x^6 + x^4 - x^3 + x^2 - 2 = 0.$$

The following points can be made.

- (i) This is a seventh-degree polynomial equation; therefore the number of real roots is 1, 3, 5 or 7.
- (ii)  $f(0)$  is negative whilst  $f(\infty) = +\infty$ , so there must be at least one positive root.

<sup>§</sup> The two roots  $\beta_1, \beta_2$  are written as  $\beta_{1,2}$ . By convention  $\beta_1$  refers to the upper symbol in  $\pm$ ,  $\beta_2$  to the lower symbol.

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Excerpt

[More information](#)

## 1.1 SIMPLE FUNCTIONS AND EQUATIONS

- (iii) The equation  $f'(x) = 0$  can be written as  $x(7x^5 + 30x^4 + 4x^2 - 3x + 2) = 0$  and thus  $x = 0$  is a root. The derivative of  $f'(x)$ , denoted by  $f''(x)$ , equals  $42x^5 + 150x^4 + 12x^2 - 6x + 2$ . That  $f'(x)$  is zero whilst  $f''(x)$  is positive at  $x = 0$  indicates (subsection 2.1.8) that  $f(x)$  has a minimum there. This, together with the facts that  $f(0)$  is negative and  $f(\infty) = \infty$ , implies that the total number of real roots to the right of  $x = 0$  must be odd. Since the total number of real roots must be odd, the number to the left must be even (0, 2, 4 or 6).

This is about all that can be deduced by *simple* analytic methods in this case, although some further progress can be made in the ways indicated in exercise 1.3.

There are, in fact, more sophisticated tests that examine the relative signs of successive terms in an equation such as (1.1), and in quantities derived from them, to place limits on the numbers and positions of roots. But they are not prerequisites for the remainder of this book and will not be pursued further here.

We conclude this section with a worked example which demonstrates that the practical application of the ideas developed so far can be both short and decisive.

► For what values of  $k$ , if any, does

$$f(x) = x^3 - 3x^2 + 6x + k = 0$$

have three real roots?

Firstly we study the equation  $f'(x) = 0$ , i.e.  $3x^2 - 6x + 6 = 0$ . This is a quadratic equation but, using (1.6), because  $6^2 < 4 \times 3 \times 6$ , it can have no real roots. Therefore, it follows immediately that  $f(x)$  has no maximum or minimum; consequently  $f(x) = 0$  cannot have more than one real root, whatever the value of  $k$ . ◀

## 1.1.2 Factorising polynomials

In the previous subsection we saw how a polynomial with  $r$  given distinct zeros  $\alpha_k$  could be constructed as the product of factors containing those zeros:

$$\begin{aligned} f(x) &= a_n(x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \cdots (x - \alpha_r)^{m_r} \\ &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \end{aligned} \quad (1.10)$$

with  $m_1 + m_2 + \cdots + m_r = n$ , the degree of the polynomial. It will cause no loss of generality in what follows to suppose that all the zeros are simple, i.e. all  $m_k = 1$  and  $r = n$ , and this we will do.

Sometimes it is desirable to be able to reverse this process, in particular when one exact zero has been found by some method and the remaining zeros are to be investigated. Suppose that we have located one zero,  $\alpha$ ; it is then possible to write (1.10) as

$$f(x) = (x - \alpha)f_1(x), \quad (1.11)$$

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Excerpt

[More information](#)

## PRELIMINARY ALGEBRA

where  $f_1(x)$  is a polynomial of degree  $n-1$ . How can we find  $f_1(x)$ ? The procedure is much more complicated to describe in a general form than to carry out for an equation with given numerical coefficients  $a_i$ . If such manipulations are too complicated to be carried out mentally, they could be laid out along the lines of an algebraic ‘long division’ sum. However, a more compact form of calculation is as follows. Write  $f_1(x)$  as

$$f_1(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \cdots + b_1x + b_0.$$

Substitution of this form into (1.11) and subsequent comparison of the coefficients of  $x^p$  for  $p = n, n-1, \dots, 1, 0$  with those in the second line of (1.10) generates the series of equations

$$\begin{aligned} b_{n-1} &= a_n, \\ b_{n-2} - \alpha b_{n-1} &= a_{n-1}, \\ b_{n-3} - \alpha b_{n-2} &= a_{n-2}, \\ &\vdots \\ b_0 - \alpha b_1 &= a_1, \\ -\alpha b_0 &= a_0. \end{aligned}$$

These can be solved successively for the  $b_j$ , starting either from the top or from the bottom of the series. In either case the final equation used serves as a check; if it is not satisfied, at least one mistake has been made in the computation – or  $\alpha$  is not a zero of  $f(x) = 0$ . We now illustrate this procedure with a worked example.

► Determine by inspection the simple roots of the equation

$$f(x) = 3x^4 - x^3 - 10x^2 - 2x + 4 = 0$$

and hence, by factorisation, find the rest of its roots.

From the pattern of coefficients it can be seen that  $x = -1$  is a solution to the equation. We therefore write

$$f(x) = (x + 1)(b_3x^3 + b_2x^2 + b_1x + b_0),$$

where

$$\begin{aligned} b_3 &= 3, \\ b_2 + b_3 &= -1, \\ b_1 + b_2 &= -10, \\ b_0 + b_1 &= -2, \\ b_0 &= 4. \end{aligned}$$

These equations give  $b_3 = 3, b_2 = -4, b_1 = -6, b_0 = 4$  (check) and so

$$f(x) = (x + 1)f_1(x) = (x + 1)(3x^3 - 4x^2 - 6x + 4).$$

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Excerpt

[More information](#)

## 1.1 SIMPLE FUNCTIONS AND EQUATIONS

We now note that  $f_1(x) = 0$  if  $x$  is set equal to 2. Thus  $x - 2$  is a factor of  $f_1(x)$ , which therefore can be written as

$$f_1(x) = (x - 2)f_2(x) = (x - 2)(c_2x^2 + c_1x + c_0)$$

with

$$\begin{aligned}c_2 &= 3, \\c_1 - 2c_2 &= -4, \\c_0 - 2c_1 &= -6, \\-2c_0 &= 4.\end{aligned}$$

These equations determine  $f_2(x)$  as  $3x^2 + 2x - 2$ . Since  $f_2(x) = 0$  is a quadratic equation, its solutions can be written explicitly as

$$x = \frac{-1 \pm \sqrt{1+6}}{3}.$$

Thus the four roots of  $f(x) = 0$  are  $-1, 2, \frac{1}{3}(-1 + \sqrt{7})$  and  $\frac{1}{3}(-1 - \sqrt{7})$ . ◀

**1.1.3 Properties of roots**

From the fact that a polynomial equation can be written in any of the alternative forms

$$\begin{aligned}f(x) &= a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0, \\f(x) &= a_n(x - \alpha_1)^{m_1}(x - \alpha_2)^{m_2} \cdots (x - \alpha_r)^{m_r} = 0, \\f(x) &= a_n(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) = 0,\end{aligned}$$

it follows that it must be possible to express the coefficients  $a_i$  in terms of the roots  $\alpha_k$ . To take the most obvious example, comparison of the constant terms (formally the coefficient of  $x^0$ ) in the first and third expressions shows that

$$a_n(-\alpha_1)(-\alpha_2) \cdots (-\alpha_n) = a_0,$$

or, using the product notation,

$$\prod_{k=1}^n \alpha_k = (-1)^n \frac{a_0}{a_n}. \quad (1.12)$$

Only slightly less obvious is a result obtained by comparing the coefficients of  $x^{n-1}$  in the same two expressions of the polynomial:

$$\sum_{k=1}^n \alpha_k = -\frac{a_{n-1}}{a_n}. \quad (1.13)$$

Comparing the coefficients of other powers of  $x$  yields further results, though they are of less general use than the two just given. One such, which the reader may wish to derive, is

$$\sum_{j=1}^n \sum_{k>j}^n \alpha_j \alpha_k = \frac{a_{n-2}}{a_n}. \quad (1.14)$$

Cambridge University Press

0521679710 - Mathematical Methods for Physics and Engineering, Third Edition

K. F. Riley, M. P. Hobson and S. J. Bence

Excerpt

[More information](#)

## PRELIMINARY ALGEBRA

In the case of a quadratic equation these root properties are used sufficiently often that they are worth stating explicitly, as follows. If the roots of the quadratic equation  $ax^2 + bx + c = 0$  are  $\alpha_1$  and  $\alpha_2$  then

$$\alpha_1 + \alpha_2 = -\frac{b}{a},$$

$$\alpha_1\alpha_2 = \frac{c}{a}.$$

If the alternative standard form for the quadratic is used,  $b$  is replaced by  $2b$  in both the equation and the first of these results.

► Find a cubic equation whose roots are  $-4, 3$  and  $5$ .

From results (1.12) – (1.14) we can compute that, arbitrarily setting  $a_3 = 1$ ,

$$-a_2 = \sum_{k=1}^3 \alpha_k = 4, \quad a_1 = \sum_{j=1}^3 \sum_{k>j}^3 \alpha_j \alpha_k = -17, \quad a_0 = (-1)^3 \prod_{k=1}^3 \alpha_k = 60.$$

Thus a possible cubic equation is  $x^3 + (-4)x^2 + (-17)x + (60) = 0$ . Of course, any multiple of  $x^3 - 4x^2 - 17x + 60 = 0$  will do just as well. ◀

## 1.2 Trigonometric identities

So many of the applications of mathematics to physics and engineering are concerned with periodic, and in particular sinusoidal, behaviour that a sure and ready handling of the corresponding mathematical functions is an essential skill. Even situations with no obvious periodicity are often expressed in terms of periodic functions for the purposes of analysis. Later in this book whole chapters are devoted to developing the techniques involved, but as a necessary prerequisite we here establish (or remind the reader of) some standard identities with which he or she should be fully familiar, so that the manipulation of expressions containing sinusoids becomes automatic and reliable. So as to emphasise the angular nature of the argument of a sinusoid we will denote it in this section by  $\theta$  rather than  $x$ .

### 1.2.1 Single-angle identities

We give without proof the basic identity satisfied by the sinusoidal functions  $\sin \theta$  and  $\cos \theta$ , namely

$$\cos^2 \theta + \sin^2 \theta = 1. \quad (1.15)$$

If  $\sin \theta$  and  $\cos \theta$  have been defined geometrically in terms of the coordinates of a point on a circle, a reference to the name of Pythagoras will suffice to establish this result. If they have been defined by means of series (with  $\theta$  expressed in radians) then the reader should refer to Euler's equation (3.23) on page 93, and note that  $e^{i\theta}$  has unit modulus if  $\theta$  is real.

---

# Index

Where the discussion of a topic runs over two consecutive pages, reference is made only to the first of these. For discussions spread over three or more pages the first and last page numbers are given; these references are usually to the major treatment of the corresponding topic. Isolated references to a topic, including those appearing on consecutive pages, are listed individually. Some long topics are split, e.g. 'Fourier transforms' and 'Fourier transforms, examples'. The letter 'n' after a page number indicates that the topic is discussed in a footnote on the relevant page.

- A, B, one-dimensional irreps, 1090, 1102, 1109  
 Abelian groups, 1044  
 absolute convergence of series, 124, 831  
 absolute derivative, 975–977  
 acceleration vector, 335  
 Adams method, 1024  
 Adams–Moulton–Bashforth, predictor–corrector scheme, 1035  
 addition rule for probabilities, 1125, 1130  
 addition theorem for spherical harmonics  
 $Y_l^m(\theta, \phi)$ , 594  
 adjoint, *see* Hermitian conjugate  
 adjoint operators, 559–564  
 adjustment of parameters, 795  
 Airy integrals, 890–894  
 $\text{Ai}(z)$ , 889  
 algebra of  
   complex numbers, 85  
   functions in a vector space, 556  
   matrices, 251  
   power series, 134  
   series, 131  
   tensors, 938–941  
   vectors, 213  
     in a vector space, 242  
     in component form, 218  
 algebraic equations, numerical methods for, *see* numerical methods for algebraic equations  
 alternating group, 1116  
 alternating series test, 130  
 ammonia molecule, symmetries of, 1042  
 Ampère's rule (law), 381, 408  
 amplitude modulation of radio waves, 444  
 amplitude-phase diagram, 914  
 analytic (regular) functions, 826  
 angle between two vectors, 221  
 angular frequency, 693n  
   in Fourier series, 419  
 angular momentum, 933, 949  
   and irreps, 1093  
   of particle system, 950–952  
   of particles, 338  
   of solid body, 396, 951  
   vector representation, 238  
 angular momentum operator  
   component, 658  
   total, 659  
 angular velocity, vector representation, 223, 238, 353  
 annihilation and creation operators, 667  
 anti-Hermitian matrices, 271  
   eigenvalues, 276–278  
     imaginary nature, 277  
   eigenvectors, 276–278  
     orthogonality, 277  
 anti-Stokes line, 905  
 anticommutativity of vector or cross product, 222  
 antisymmetric functions, 416  
   and Fourier series, 419  
   and Fourier transforms, 445  
 antisymmetric matrices, 270

## INDEX

- general properties, *see* anti-Hermitian matrices
- antisymmetric tensors, 938, 941
- antithetic variates, in Monte Carlo methods, 1014
- aperture function, 437
- approximately equal  $\approx$ , definition, 132
- arbitrary parameters for ODE, 469
- arc length of  
plane curves, 73  
space curves, 341
- arccosech, arccosh, arccoth, arcsech, arcsinh, arctanh, *see* hyperbolic functions, inverses
- Archimedean upthrust, 396, 410
- area element in  
Cartesian coordinates, 188  
plane polars, 202
- area of  
circle, 71  
ellipse, 71, 207  
parallelogram, 223  
region, using multiple integrals, 191–193  
surfaces, 346  
as vector, 393–395, 408
- area, maximal enclosure, 779
- arg, argument of a complex number, 87
- Argand diagram, 84, 825
- argument, principle of the, 880
- arithmetic series, 117
- arithmetic-geometric series, 118
- arrays, *see* matrices
- associated Laguerre equation, 535, 621–624  
as example of Sturm–Liouville equation, 566, 622  
natural interval, 567, 622
- associated Laguerre polynomials  $L_n^m(x)$ , 621  
as special case of confluent hypergeometric function, 634  
generating function, 623  
orthogonality, 622  
recurrence relations, 624  
Rodrigues' formula, 622
- associated Legendre equation, 535, 587–593, 733, 768  
general solution, 588  
as example of Sturm–Liouville equation, 566, 590, 591  
general solution, 588  
natural interval, 567, 590, 591
- associated Legendre functions, 587–593  
of first kind  $P_\ell^m(x)$ , 588, 733, 768  
generating function, 592  
normalisation, 590  
orthogonality, 590, 591  
recurrence relations, 592  
Rodrigues' formula, 588  
of second kind  $Q_\ell^m(x)$ , 588
- associative law for addition  
in a vector space of finite dimensionality, 242  
in a vector space of infinite dimensionality, 556  
of complex numbers, 86  
of matrices, 251  
of vectors, 213
- convolution, 447, 458
- group operations, 1043
- linear operators, 249
- multiplication  
of a matrix by a scalar, 251  
of a vector by a scalar, 214  
of complex numbers, 88  
of matrices, 253
- multiplication by a scalar  
in a vector space of finite dimensionality, 242  
in a vector space of infinite dimensionality, 556
- atomic orbitals, 1115  
 $d$ -states, 1106, 1108, 1114  
 $p$ -states, 1106  
 $s$ -states, 1144
- auto-correlation functions, 450
- automorphism, 1061
- auxiliary equation, 493  
repeated roots, 493
- average value, *see* mean value
- axial vectors, 949
- backward differences, 1019
- basis functions  
for linear least squares estimation, 1273  
in a vector space of infinite dimensionality, 556  
of a representation, 1078  
change in, 1084, 1087, 1092
- basis vectors, 217, 243, 929, 1078
- derivatives, 965–968  
Christoffel symbol  $\Gamma_{ij}^k$ , 965  
for particular irrep, 1106–1108, 1116  
linear dependence and independence, 217  
non-orthogonal, 245  
orthonormal, 244  
required properties, 217
- Bayes' theorem, 1132
- Bernoulli equation, 477
- Bessel correction to variance estimate, 1248
- Bessel equation, 535, 602–607, 614, 615  
as example of Sturm–Liouville equation, 566  
natural interval, 608
- Bessel functions  $J_\nu(x)$ , 602–614, 729, 738  
as special case of confluent hypergeometric function, 634  
generating function, 613  
graph of, 606  
integral relationships, 610  
integral representation, 613–614  
orthogonality, 608–611

## INDEX

- recurrence relations, 611–612  
 second kind  $Y_\nu(x)$ , 607  
   graph of, 607  
   series, 604  
    $\nu = 0$ , 606  
    $\nu = \pm 1/2$ , 605  
   spherical  $j_\nu(x)$ , 615, 741  
   zeros of, 729, 739  
 Bessel inequality, 246, 559  
 best unbiased estimator, 1232  
 beta function, 638  
 bias of estimator, 1231  
 bilinear transformation, general, 110  
 binary chopping, 990  
 binomial coefficient  ${}^nC_k$ , 27–30, 1135–1137  
   elementary properties, 26  
   identities, 27  
   in Leibnitz' theorem, 49  
   negative  $n$ , 29  
   non-integral  $n$ , 29  
 binomial distribution  $\text{Bin}(n, p)$ , 1168–1171  
   and Gaussian distribution, 1185  
   and Poisson distribution, 1174, 1177  
   mean and variance, 1171  
   MGF, 1170  
   recurrence formula, 1169  
 binomial expansion, 25–30, 140  
 binormal to space curves, 342  
 birthdays, different, 1134  
 bivariate distributions, 1196–1207  
   conditional, 1198  
   continuous, 1197  
   correlation, 1200–1207  
     and independence, 1200  
     matrix, 1203–1207  
     positive and negative, 1200  
     uncorrelated, 1200  
   covariance, 1200–1207  
     matrix, 1203  
   expectation (mean), 1199  
   independent, 1197, 1200  
   marginal, 1198  
   variance, 1200  
 Boltzmann distribution, 171  
 bonding in molecules, 1103, 1105–1108  
 Born approximation, 149, 575  
 Bose–Einstein statistics, 1138  
 boundary conditions  
   and characteristics, 700  
   and Laplace equation, 764, 766  
   for Green's functions, 512, 514–516  
     inhomogeneous, 515  
   for ODE, 468, 470, 501  
   for PDE, 681, 685–687  
   for Sturm–Liouville equations, 564  
   homogeneous and inhomogeneous, 685, 723,  
     752, 754  
   superposition solutions, 718–724  
   types, 702–705  
 bra vector  $\langle \psi |$ , 649  
 brachistochrone problem, 784  
 Bragg formula, 237  
 branch cut, 835  
 branch points, 835  
 Bromwich integral, 884  
 bulk modulus, 980  
 calculus of residues, *see* zeros of a function of a  
   complex variable *and* contour integration  
 calculus of variations  
   constrained variation, 785–787  
   estimation of ODE eigenvalues, 790  
   Euler–Lagrange equation, 776  
   Fermat's principle, 787  
   Hamilton's principle, 788  
   higher-order derivatives, 782  
   several dependent variables, 782  
   several independent variables, 782  
   soap films, 780  
   variable end-points, 782–785  
 calculus, elementary, 41–76  
 cancellation law in a group, 1046  
 canonical form, for second-order ODE, 516  
 card drawing, *see* probability  
 carrier frequency of radio waves, 445  
 Cartesian coordinates, 217  
 Cartesian tensors, 930–955  
   algebra, 938–941  
   contraction, 939  
   definition, 935  
   first-order, 932–935  
     from scalar, 934  
   general order, 935–954  
   integral theorems, 954  
   isotropic, 944–946  
   physical applications, 934, 939–941, 950–954  
   second-order, 935–954, 968  
   symmetry and antisymmetry, 938  
   tensor fields, 954  
   zero-order, 932–935  
     from vector, 935  
 Cartesian tensors, particular  
   conductivity, 952  
   inertia, 951  
   strain, 953  
   stress, 953  
   susceptibility, 952  
 catenary, 781, 787  
 Cauchy  
   boundary conditions, 702  
   distribution, 1152  
   inequality, 853  
   integrals, 851–853  
   product, 131  
   root test, 129, 831  
   theorem, 849  
 Cauchy–Riemann relations, 827–830, 849, 873,  
   875  
   in terms of  $z$  and  $z^*$ , 829  
 central differences, 1019

## INDEX

- central limit theorem, 1194–1196  
 central moments, *see* moments, central  
 centre of a group, 1069  
 centre of mass, 195  
   of hemisphere, 195  
   of semicircular lamina, 197  
 centroid, 195  
   of plane area, 195  
   of plane curve, 197  
   of triangle, 216  
 CF, *see* complementary function  
 chain rule for functions of  
   one real variable, 46  
   several real variables, 157  
 change of basis, *see* similarity transformations  
 change of variables  
   and coordinate systems, 158–160  
   in multiple integrals, 199–207  
     evaluation of Gaussian integral, 202–204  
     general properties, 206  
   in RVD, 1150–1157  
 character tables, 1093  
    $3m$  or  $32$  or  $C_{3v}$  or  $S_3$ , 1093, 1097, 1108, 1110, 1117  
    $4mm$  or  $C_{4v}$  or  $D_4$ , 1102, 1106, 1108, 1113  
    $A_4$ , 1116  
    $D_5$ , 1116  
    $S_4$  or  $432$  or  $O$ , 1114, 1115  
    $43m$  or  $T_d$ , 1115  
   construction of, 1100–1102  
   quaternion, 1113  
 characteristic equation, 280  
   normal mode form, 319  
   of recurrence relation, 499  
 characteristic functions, *see* moment generating functions (MGFs)  
 characteristics  
   and boundary curves, 700  
   multiple intersections, 700, 705  
   and the existence of solutions, 699–705  
   first-order equations, 699  
   second-order equations, 703  
     and equation type, 703  
 characters, 1092–1096, 1100–1102  
   and conjugacy classes, 1092, 1095  
   counting irreps, 1095  
   definition, 1092  
   of product representation, 1104  
   orthogonality properties, 1094, 1102  
   summation rules, 1097  
 charge (point), Dirac  $\delta$ -function representation, 441  
 charged particle in electromagnetic fields, 370  
 Chebyshev equation, 535, 595–602  
   as example of Sturm–Liouville equation, 566, 599  
   general solution, 597  
   natural interval, 567, 599  
   polynomial solutions, 552  
 Chebyshev functions, 595–602  
 Chebyshev polynomials  
   of first kind  $T_n(x)$ , 596  
     as special case of hypergeometric function, 631  
     generating function, 601  
     graph of, 597  
     normalisation, 600  
     orthogonality, 599  
     recurrence relations, 601  
     Rodrigues' formula, 599  
   of second kind  $U_n(x)$ , 597  
     generating function, 601  
     graph of, 598  
     normalisation, 600  
     orthogonality, 599  
     Rodrigues' formula, 599  
 chi-squared ( $\chi^2$ ) distribution, 1192  
   and goodness of fit, 1297  
   and likelihood-ratio test, 1283, 1292  
   and multiple estimators, 1243  
   percentage points tabulation, 1244  
   test for correlation, 1301  
 Cholesky separation, 313  
 Christoffel symbol  $\Gamma^k_{ij}$ , 965–968  
   from metric tensor, 966, 973  
 circle  
   area of, 71  
   equation for, 16  
 circle of convergence, 831  
 Clairaut equation, 483  
 classes and equivalence relations, 1064  
 closure of a group, 1043  
 closure property of eigenfunctions of an Hermitian operator, 563  
 cofactor of a matrix element, 259  
 column matrix, 250  
 column vector, 250  
 combinations (probability), 1133–1139  
 common ratio in geometric series, 117  
 commutation law for group elements, 1044  
 commutative law for  
   addition  
     in a vector space of finite dimensionality, 242  
     in a vector space of infinite dimensionality, 556  
   of complex numbers, 86  
   of matrices, 251  
   of vectors, 213  
   complex scalar or dot product, 222  
 convolution, 447, 458  
 inner product, 244  
 multiplication  
   of a vector by a scalar, 214  
   of complex numbers, 88  
   scalar or dot product, 220  
 commutator  
   of two matrices, 309  
   of two operators, 653, 656  
 comparison test, 125

## INDEX

- complement, 1121  
 probability for, 1125
- complementary equation, 490
- complementary error function, 640
- complementary function (CF), 491  
 for ODE, 492  
 partially known, 506  
 repeated roots of auxiliary equation, 493
- completeness of  
 basis vectors, 243  
 eigenfunctions of an Hermitian operator, 560, 563  
 eigenvectors of a normal matrix, 275  
 spherical harmonics  $Y_\ell^m(\theta, \phi)$ , 594
- completing the square  
 as a means of integration, 66  
 for quadratic equations, 35  
 for quadratic forms, 1206  
 to evaluate Gaussian integral, 436, 749
- complex conjugate  
 $z^*$ , of complex number, 89–91, 829  
 of a matrix, 256–258  
 of scalar or dot product, 222  
 properties of, 90
- complex exponential function, 92, 833
- complex Fourier series, 424
- complex integrals, 845–849, *see also* zeros of a function of a complex variable *and* contour integration  
 Airy integrals, 890–894  
 Cauchy integrals, 851–853  
 Cauchy's theorem, 849  
 definition, 845  
 Jordan's lemma, 864  
 Morera's theorem, 851  
 of  $z^{-1}$ , 846  
 principal value, 864  
 residue theorem, 858–860  
 WKB methods, 895–905
- complex logarithms, 99, 834  
 principal value of, 100, 834
- complex numbers, 83–114  
 addition and subtraction of, 85  
 applications to differentiation and integration, 101  
 argument of, 87  
 associativity of  
 addition, 86  
 multiplication, 88  
 commutativity of  
 addition, 86  
 multiplication, 88  
 complex conjugate of, *see* complex conjugate  
 components of, 84  
 de Moivre's theorem, *see* de Moivre's theorem  
 division of, 91, 94  
 from roots of polynomial equations, 83  
 imaginary part of, 83  
 modulus of, 87  
 multiplication of, 88, 94  
 as rotation in the Argand diagram, 88  
 notation, 84  
 polar representation of, 92–95  
 real part of, 83  
 trigonometric representation of, 93
- complex potentials, 871–876  
 and fluid flow, 873  
 equipotentials and field lines, 872  
 for circular and elliptic cylinders, 876  
 for parallel cylinders, 921  
 for plates, 877–879, 921  
 for strip, 921  
 for wedges, 878  
 under conformal transformations, 876–879
- complex power series, 133
- complex powers, 99
- complex variables, *see* functions of a complex variable *and* power series in a complex variable *and* complex integrals
- components  
 of a complex number, 84  
 of a vector, 217  
 in a non-orthogonal basis, 234  
 uniqueness, 243
- conditional (constrained) variation, 785–787
- conditional convergence, 124
- conditional distributions, 1198
- conditional probability, *see* probability, conditional
- cone  
 surface area of, 74  
 volume of, 75
- confidence interval, 1236
- confidence region, 1241
- confluence process, 634
- confluent hypergeometric equation, 535, 633  
 as example of Sturm–Liouville equation, 566  
 general solution, 633
- confluent hypergeometric functions, 633
- contiguous relations, 635
- integral representation, 634
- recurrence relations, 635
- special cases, 634
- conformal transformations (mappings), 839–879  
 applications, 876–879  
 examples, 842–844  
 properties, 839–842  
 Schwarz–Christoffel transformation, 843
- congruence, 1065
- conic sections, 15  
 eccentricity, 17  
 parametric forms, 17  
 standard forms, 16
- conjugacy classes, 1068–1070  
 element in a class by itself, 1068
- conjugate roots of polynomial equations, 99
- connectivity of regions, 383
- conservative fields, 387–389  
 necessary and sufficient conditions, 387–389  
 potential (function), 389

## INDEX

- consistency, of estimator, 1230  
 constant coefficients in ODE, 492–503  
   auxiliary equation, 493  
 constants of integration, 62, 468  
 constrained variation, 785–787  
 constraints, stationary values under, *see*  
   Lagrange undetermined multipliers  
 continuity correction for discrete RV, 1186  
 continuity equation, 404  
 contour integration, 861–867, 887  
   infinite integrals, 862–867  
   inverse Laplace transforms, 884–887  
   residue theorem, 858–867  
   sinusoidal functions, 861  
   summing series, 882  
 contraction of tensors, 939  
 contradiction, proof by, 32–34  
 contravariant  
   basis vectors, 961  
   derivative, 965  
   components of tensor, 956  
   definition, 961  
 control variates, in Monte Carlo methods, 1013  
 convergence of infinite series, 831  
   absolute, 124, 831  
   complex power series, 133  
   conditional, 124  
   necessary condition, 125  
   power series, 132  
   under various manipulations, *see* power  
   series, manipulation  
 ratio test, 832  
 rearrangement of terms, 124  
 tests for convergence, 125–131  
   alternating series test, 130  
   comparison test, 125  
   grouping terms, 129  
   integral test, 128  
   quotient test, 127  
   ratio comparison test, 127  
   ratio test (D'Alembert), 126, 132  
   root test (Cauchy), 129, 831  
 convergence of numerical iteration schemes,  
 992–994  
 convolution  
   Fourier transforms, *see* Fourier transforms,  
   convolution  
   Laplace transforms, *see* Laplace transforms,  
   convolution  
 convolution theorem  
   Fourier transforms, 448  
   Laplace transforms, 457  
 coordinate geometry, 15–18  
   conic sections, 15  
   straight line, 15  
 coordinate systems, *see* Cartesian, curvilinear,  
   cylindrical polar, plane polar *and* spherical  
   polar coordinates  
 coordinate transformations  
   and integrals, *see* change of variables  
   and matrices, *see* similarity transformations  
   general, 960–965  
   relative tensors, 963  
   tensor transformations, 962  
   weight, 964  
   orthogonal, 932  
 coplanar vectors, 225  
 Cornu spiral, 914  
 correlation functions, 449–451  
   auto-correlation, 450  
   cross-correlation, 449  
   energy spectrum, 450  
   Parseval's theorem, 451  
   Wiener–Kinchin theorem, 450  
 correlation matrix, of sample, 1229  
 correlation of bivariate distributions, 1200–1207  
 correlation of sample data, 1229  
 correlation, chi-squared test, 1301  
 correspondence principle in quantum mechanics,  
 1215  
 cosets and congruence, 1065  
 cosh, hyperbolic cosine, 102, 833, *see also*  
   hyperbolic functions  
 cosine,  $\cos(x)$   
   in terms of exponential functions, 102  
   Maclaurin series for, 140  
   orthogonality relations, 417  
 counting irreps, *see* characters, counting irreps  
 coupled pendulums, 329, 331  
 covariance matrix  
   of linear least squares estimators, 1274  
   of sample, 1229  
 covariance of bivariate distributions, 1200–1207  
 covariance of sample data, 1229  
 covariant  
   basis vector, 961  
   derivative, 965  
   components of tensor, 956  
   definition, 961  
   derivative, 968  
   of scalar, 971  
   semi-colon notation, 969  
   differentiation, 968–971  
 CPF, *see* probability functions, cumulative  
 Cramér–Rao (Fisher's) inequality, 1232, 1233  
 Cramer determinant, 299  
 Cramer's rule, 299  
 cross product, *see* vector product  
 cross-correlation functions, 449  
 crystal lattice, 148  
 crystal point groups, 1082  
 cube roots of unity, 98  
 cube, rotational symmetries of, 1114  
 cumulants, 1166  
 curl of a vector field, 353  
   as a determinant, 353  
   as integral, 398, 400  
   curl curl, 356  
   in curvilinear coordinates, 368  
   in cylindrical polars, 360

## INDEX

- in spherical polars, 362  
Stoke's theorem, 406–409  
tensor form, 974
- current-carrying wire, magnetic potential, 729
- curvature, 52–55  
circle of, 53  
of a function, 52  
of space curves, 342  
radius of, 53
- curves, *see* plane curves *and* space curves
- curvilinear coordinates, 364–369  
basis vectors, 364  
length and volume elements, 365  
scale factors, 364  
surfaces and curves, 364  
tensors, 955–977  
vector operators, 367–369
- cut plane, 865
- cycle notation for permutations, 1057
- cyclic groups, 1061, 1098
- cyclic relation for partial derivatives, 157
- cycloid, 370, 785
- cylinders, conducting, 874, 876
- cylindrical polar coordinates, 357–361  
area element, 360  
basis vectors, 358  
Laplace equation, 728–731  
length element, 360  
vector operators, 357–361  
volume element, 360
- $\delta$ -function (Dirac), *see* Dirac  $\delta$ -function
- $\delta_{ij}$ ,  $\delta_i^j$ , Kronecker delta, tensor, *see* Kronecker delta,  $\delta_{ij}$ ,  $\delta_i^j$ , tensor
- D'Alembert's ratio test, 126, 832  
in convergence of power series, 132
- D'Alembert's solution to wave equation, 694
- damped harmonic oscillators, 239  
and Parseval's theorem, 451
- data modelling, maximum-likelihood, 1255
- de Broglie relation, 436, 709, 768
- de Moivre's theorem, 95, 861  
applications, 95–99  
finding the  $n$ th roots of unity, 97  
solving polynomial equations, 98  
trigonometric identities, 95–97
- deconvolution, 449
- defective matrices, 278, 311
- degeneracy  
breaking of, 1111–1113  
of normal modes, 1110
- degenerate (separable) kernel, 807
- degenerate eigenvalues, 275, 282
- degree  
of ODE, 468  
of polynomial equation, 2
- del  $\nabla$ , *see* gradient operator (grad)
- del squared  $\nabla^2$  (Laplacian), 352, 676  
as integral, 400
- in curvilinear coordinates, 368
- in cylindrical polar coordinates, 360
- in polar coordinates, 725
- in spherical polar coordinates, 362, 741  
tensor form, 973
- delta function (Dirac), *see* Dirac  $\delta$ -function
- dependent random variables, 1196–1205
- derivative, *see also* differentiation  
absolute, 975–977  
covariant, 968  
Fourier transform of, 444  
Laplace transform of, 455  
normal, 350  
of basis vectors, 336  
of composite vector expressions, 337  
of function of a complex variable, 825  
of function of a function, 46  
of hyperbolic functions, 106–109  
of products, 44–46, 48–50  
of quotients, 47  
of simple functions, 44  
of vectors, 334  
ordinary, first, second and  $n$ th, 42  
partial, *see* partial differentiation  
total, 154
- derivative method for second series solution of ODE, 545–548
- determinant form  
and  $\epsilon_{ijk}$ , 942  
for curl, 353
- determinants, 259–263  
adding rows or columns, 262  
and singular matrices, 263  
as product of eigenvalues, 287  
evaluation  
using  $\epsilon_{ijk}$ , 942  
using Laplace expansion, 259
- identical rows or columns, 262
- in terms of cofactors, 259
- interchanging two rows or two columns, 262
- Jacobian representation, 201, 205, 207  
notation, 259  
of Hermitian conjugate matrices, 262  
of order three, in components, 260  
of transpose matrices, 261  
product rule, 262  
properties, 261–263, 978  
relationship with rank, 267  
removing factors, 262  
secular, 280
- diagonal matrices, 268
- diagonalisation of matrices, 285–288  
normal matrices, 286  
properties of eigenvalues, 287  
simultaneous, 331
- diamond, unit cell, 234
- die throwing, *see* probability
- difference method for summation of series, 119
- difference schemes for differential equations, 1020–1023, 1030–1032

## INDEX

- difference, finite, *see* finite differences  
 differentiable  
   function of a complex variable, 825–827  
   function of a real variable, 42  
 differential  
   definition, 43  
   exact and inexact, 155  
   of vector, 338, 344  
   total, 154  
 differential equations, *see* ordinary differential  
   equations *and* partial differential equations  
 differential equations, particular  
   associated Laguerre, 535, 566, 621–624  
   associated Legendre, 535, 566, 587–593  
   Bernoulli, 477  
   Bessel, 535, 566, 602–607, 614  
   Chebyshev, 535, 566, 595–602  
   Clairaut, 483  
   confluent hypergeometric, 535, 566, 633  
   diffusion, 678, 695–698, 716, 723, 1032  
   Euler, 504  
   Euler–Lagrange, 776  
   Helmholtz, 737–741  
   Hermite, 535, 566, 624–628  
   hypergeometric, 535, 566, 628–632  
   Lagrange, 789  
   Laguerre, 535, 566, 616–621  
   Laplace, 679, 690, 717, 718, 1031  
   Legendre, 534, 535, 566, 577–586  
   Legendre linear, 503–505  
   Poisson, 679, 744–746  
   Schrödinger, 679, 741, 768, 795  
   simple harmonic oscillator, 535, 566  
   Sturm–Liouville, 790  
   wave, 676, 689, 693–695, 714, 737, 790  
 differential operators, *see* linear differential  
   operator  
 differentiation, *see also* derivative  
   as gradient, 42  
   as rate of change, 41  
   chain rule, 46  
   covariant, 968–971  
   from first principles, 41–44  
   implicit, 47  
   logarithmic, 48  
   notation, 43  
   of Fourier series, 424  
   of integrals, 178  
   of power series, 135  
   partial, *see* partial differentiation  
   product rule, 44–46, 48–50  
   quotient rule, 47  
   theorems, 55–57  
   using complex numbers, 101  
 diffraction, *see* Fraunhofer diffraction  
 diffusion equation, 678, 688, 695–698  
   combination of variables, 696–698  
   integral transforms, 747  
   numerical methods, 1032  
   separation of variables, 716  
   simple solution, 696  
   superposition, 723  
 diffusion of solute, 678, 696, 747  
 dihedral group, 1113, 1116  
 dimension of irrep, 1088  
 dimensionality of vector space, 243  
 dipole matrix elements, 208, 1108, 1115  
 dipole moments of molecules, 1077  
 Dirac  $\delta$ -function, 355, 405, 439–443  
   and convolution, 447  
   and Green's functions, 511, 512  
   as limit of various distributions, 443  
   as sum of harmonic waves, 442  
   definition, 439  
   Fourier transform of, 443  
   impulses, 441  
   point charges, 441  
   properties, 439  
   reality of, 443  
   relation to Fourier transforms, 442  
   relation to Heaviside (unit step) function, 441  
   three-dimensional, 441, 452  
 Dirac notation, 648  
 direct product, of groups, 1072  
 direct sum  $\oplus$ , 1086  
 direction cosines, 221  
 Dirichlet boundary conditions, 702, 852n  
   Green's functions, 754, 756–765  
   method of images, 758–765  
 Dirichlet conditions, for Fourier series, 415  
 disc, moment of inertia, 208  
 discontinuous functions and Fourier series,  
   420–422  
 discrete Fourier transforms, 462  
 disjoint events, *see* mutually exclusive events  
 displacement kernel, 809  
 distance from a  
   line to a line, 231  
   line to a plane, 232  
   point to a line, 229  
   point to a plane, 230  
 distributive law for  
   addition of matrix products, 254  
   convolution, 447, 458  
   inner product, 244  
   linear operators, 249  
   multiplication  
     of a matrix by a scalar, 251  
     of a vector by a complex scalar, 222  
     of a vector by a scalar, 214  
   multiplication by a scalar  
     in a vector space of finite dimensionality,  
       242  
     in a vector space of infinite dimensionality,  
       556  
   scalar or dot product, 220  
   vector or cross product, 222  
 div, divergence of vector fields, 352  
   as integral, 398  
   in curvilinear coordinates, 367

## INDEX

- in cylindrical polars, 360
- in spherical polars, 362
- tensor form, 972
- divergence theorem
  - for tensors, 954
  - for vectors, 401
  - in two dimensions, 384
  - physical applications, 404
  - related theorems, 403
- division axiom in a group, 1046
- division of complex numbers, 91
- dominant term, in Stokes phenomenon, 904
- dot product, *see* scalar product
- double integrals, *see* multiple integrals
- drumskin, *see* membrane
- dual tensors, 949
- dummy variable, 61
- $\epsilon_{ijk}$ , Levi-Civita symbol, tensor, 941–946
  - and determinant, 942
  - identities, 943
  - isotropic, 945
  - vector products, 942
  - weight, 964
- $e^x$ , *see* exponential function
- E, two-dimensional irrep, 1090, 1102, 1108
- eccentricity, of conic sections, 17
- efficiency, of estimator, 1231
- eigenequation for differential operators, 554
  - more general form, 555, 571–573
- eigenfrequencies, 319
  - estimation using Rayleigh–Ritz method, 327–329
- eigenfunctions
  - completeness for Hermitian operators, 560, 563
  - construction of a real set for an Hermitian operator, 563
  - definition, 555
  - normalisation for Hermitian operators, 562
  - of integral equations, 817
  - of simple harmonic oscillators, 555
  - orthogonality for Hermitian operators, 561–563
- eigenvalues, 272–282, *see* Hermitian operators
  - characteristic equation, 280
  - continuous and discrete, 650
  - definition, 272
  - degenerate, 282
  - determination, 280–282
  - estimation for ODE, 790
  - estimation using Rayleigh–Ritz method, 327–329
  - notation, 273
  - of anti-Hermitian matrices, *see* anti-Hermitian matrices
  - of Fredholm equations, 808
  - of general square matrices, 278
  - of Hermitian matrices, *see* Hermitian matrices
  - of integral equations, 808, 816
  - of linear differential operators
    - adjustment of parameters, 795
    - definition, 555
    - error in estimate of, 793
    - estimation, 790–796
    - higher eigenvalues, 793, 800
    - simple harmonic oscillator, 555
  - of linear operators, 272
  - of normal matrices, 273–276
  - of representative matrices, 1100
  - of unitary matrices, 278
  - under similarity transformation, 287
- eigenvectors, 272–282
  - characteristic equation, 280
  - definition, 272
  - determination, 280–282
  - normalisation condition, 273
  - notation, 273
  - of anti-Hermitian matrices, *see* anti-Hermitian matrices
  - of commuting matrices, 278
  - of general square matrices, 278
  - of Hermitian matrices, *see* Hermitian matrices
  - of linear operators, 272
  - of normal matrices, 273–276
  - of unitary matrices, 278
  - stationary properties for quadratic and Hermitian forms, 290
- Einstein relation, 436, 709, 768
- elastic deformations, 953
- electromagnetic fields
  - flux, 395
  - Maxwell's equations, 373, 408, 979
- electrostatic fields and potentials
  - charged split sphere, 735
  - conducting cylinder in uniform field, 876
  - conducting sphere in uniform field, 734
  - from charge density, 745, 758
  - from complex potential, 873
  - infinite charged plate, 759, 877
  - infinite wedge with line charge, 878
  - infinite charged wedge, 877
  - of line charges, 761, 872
  - semi-infinite charged plate, 877
  - sphere with point charge, 764
- ellipse
  - area of, 71, 207, 385
  - as section of quadratic surface, 292
  - equation for, 16
- ellipsoid, volume of, 207
- elliptic PDE, 687, 690
- empty event  $\emptyset$ , 1121
- end-points for variations
  - contributions from, 782
  - fixed, 777
  - variable, 782–785
- energy levels of
  - particle in a box, 768
  - simple harmonic oscillator, 642

## INDEX

- energy spectrum and Fourier transforms, 450, 451
- entire functions, 832n
- envelopes, 173–175  
equations of, 174  
to a family of curves, 173
- epimorphism, 1061
- equilateral triangle, symmetries of, 1047, 1052, 1081, 1110
- equivalence relations, 1064–1066, 1068  
and classes, 1064  
congruence, 1065–1067  
examples, 1070
- equivalence transformations, *see* similarity transformations
- equivalent representations, 1084–1086, 1099
- error function,  $\text{erf}(x)$ , 640, 697, 748  
as special case of confluent hypergeometric function, 634
- error terms  
in Fourier series, 430  
in Taylor series, 139
- errors, first and second kind, 1280
- essential singularity, 838, 856
- estimation of eigenvalues  
linear differential operator, 792–795  
Rayleigh–Ritz method, 327–329
- estimators (statistics), 1229  
best unbiased, 1232  
bias, 1231  
central confidence interval, 1237  
confidence interval, 1236  
confidence limits, 1236  
confidence region, 1241  
consistency, 1230  
efficiency, 1231  
maximum-likelihood, 1256  
minimum-variance, 1232  
standard error, 1234
- Euler equation  
differential, 504, 522  
trigonometric, 93
- Euler method, numerical, 1021
- Euler–Lagrange equation, 776  
special cases, 777–781
- even functions, *see* symmetric functions
- events, 1120  
complement of, 1121  
empty  $\emptyset$ , 1121  
intersection of  $\cap$ , 1120  
mutually exclusive, 1129  
statistically independent, 1129  
union of  $\cup$ , 1121
- exact differentials, 155
- exact equations, 472, 505  
condition for, 472  
non-linear, 519
- expectation values, *see* probability distributions, mean
- exponential distribution, 1190  
from Poisson, 1190  
MGF, 1191
- exponential function  
Maclaurin series for, 140  
of a complex variable, 92, 833  
relation with hyperbolic functions, 102
- $F$ -distribution (Fisher), 1290–1296  
critical points table, 1295  
logarithmic form, 1296
- Fabry–Pérot interferometer, 146
- factorial function, general, 636
- factorisation, of a polynomial equation, 7
- faithful representation, 1083, 1098
- Fermat’s principle, 787, 798
- Fermi–Dirac statistics, 1138
- Fibonacci series, 525
- field lines and complex potentials, 872
- fields  
conservative, 387–389  
scalar, 347  
tensor, 954  
vector, 347
- fields, electrostatic, *see* electrostatic fields and potentials
- fields, gravitational, *see* gravitational fields and potentials
- finite differences, 1019  
central, 1019  
for differential equations, 1020–1023  
forward and backward, 1019  
from Taylor series, 1019, 1026  
schemes for differential equations, 1030–1032
- finite groups, 1043
- first law of thermodynamics, 176
- first-order differential equations, *see* ordinary differential equations
- Fisher distribution, *see*  $F$ -distribution (Fisher)
- Fisher matrix, 1241, 1268
- Fisher’s inequality, 1232, 1233
- fluids  
Archimedean upthrust, 396, 410  
complex velocity potential, 873  
continuity equation, 404  
cylinder in uniform flow, 874  
flow, 873  
flux, 395, 875  
irrotational flow, 353  
sources and sinks, 404, 873  
stagnation points, 873  
velocity potential, 409, 679  
vortex flow, 408, 874
- forward differences, 1019
- Fourier cosine transforms, 446
- Fourier series, 415–432  
and separation of variables, 719–722, 724  
coefficients, 417–419, 425  
complex, 424  
differentiation, 424  
Dirichlet conditions, 415

## INDEX

- discontinuous functions, 420–422  
 error term, 430  
 integration, 424  
 non-periodic functions, 422–424  
 orthogonality of terms, 417  
   complex case, 425  
 Parseval's theorem, 426  
 standard form, 417  
 summation of series, 427  
 symmetry considerations, 419  
 uses, 415
- Fourier series, examples  
   square-wave, 418  
    $x$ , 424, 425  
    $x^2$ , 422  
    $x^3$ , 424
- Fourier sine transforms, 445
- Fourier transforms, 433–453  
   as generalisation of Fourier series, 433–435  
   convolution, 446–449  
     and the Dirac  $\delta$ -function, 447  
     associativity, commutativity, distributivity, 447  
     definition, 447  
     resolution function, 446  
   convolution theorem, 448  
   correlation functions, 449–451  
   cosine transforms, 446  
   deconvolution, 449  
   definition, 435  
   discrete, 462  
   evaluation using convolution theorem, 448  
   for integral equations, 809–812  
   for PDE, 749–751  
   Fourier-related (conjugate) variables, 436  
   in higher dimensions, 451–453  
   inverse, definition, 435  
   odd and even functions, 445  
   Parseval's theorem, 450  
   properties: differentiation, exponential multiplication, integration, scaling, translation, 444  
   relation to Dirac  $\delta$ -function, 442  
   sine transforms, 445
- Fourier transforms, examples  
   convolution, 448  
   damped harmonic oscillator, 451  
   Dirac  $\delta$ -function, 443  
   exponential decay function, 435  
   Gaussian (normal) distribution, 435  
   rectangular distribution, 442  
   spherically symmetric functions, 452  
   two narrow slits, 448  
   two wide slits, 438, 448
- Fourier's inversion theorem, 435
- Fraunhofer diffraction, 437–439  
   diffraction grating, 461  
   two narrow slits, 448  
   two wide slits, 438, 448
- Fredholm integral equations, 805  
   eigenvalues, 808  
   operator form, 806  
   with separable kernel, 807
- Fredholm theory, 815
- Frenet–Serret formulae, 343
- Fresnel integrals, 913
- Frobenius series, 539
- Fuch's theorem, 539
- function of a matrix, 255
- functional, 776
- functions of a complex variable, 825–839,  
   853–858  
   analyticity, 826  
   behaviour at infinity, 839  
   branch points, 835  
   Cauchy integrals, 851–853  
   Cauchy–Riemann relations, 827–830  
   conformal transformations, 839–844  
   derivative, 825  
   differentiation, 825–830  
   identity theorem, 854  
   Laplace equation, 829, 871  
   Laurent expansion, 855–858  
   multivalued and branch cuts, 835–837, 885  
   particular functions, 832–835  
   poles, 837  
   power series, 830–832  
   real and imaginary parts, 825, 830  
   singularities, 826, 837–839  
   Taylor expansion, 853–855  
   zeros, 839, 879–882
- functions of one real variable  
   decomposition into even and odd functions, 416  
   differentiation of, 41–50  
   Fourier series, *see* Fourier series  
   integration of, 59–72  
   limits, *see* limits  
   maxima and minima of, 50–52  
   stationary values of, 50–52  
   Taylor series, *see* Taylor series
- functions of several real variables  
   chain rule, 157  
   differentiation of, 151–179  
   integration of, *see* multiple integrals,  
     evaluation  
   maxima and minima, 162–167  
   points of inflection, 162–167  
   rates of change, 153–155  
   saddle points, 162–167  
   stationary values, 162–167  
   Taylor series, 160–162
- fundamental solution, 757
- fundamental theorem of  
   algebra, 83, 85, 868  
   calculus, 61  
   complex numbers, *see* de Moivre's theorem
- gamma distribution, 1153, 1191
- gamma function

## INDEX

- as general factorial function, 636
- definition and properties, 636
- graph of, 637
- Gauss's theorem, 765
- Gauss–Seidel iteration, 996–998
- Gaussian (normal) distribution  $N(\mu, \sigma^2)$ , 1179–1189
  - and binomial distribution, 1185
  - and central limit theorem, 1195
  - and Poisson distribution, 1187
  - continuity correction, 1186
  - CPF, 1018, 1181
    - tabulation, 1182
  - Fourier transform, 435
  - integration with infinite limits, 202–204
  - mean and variance, 1180–1184
  - MGF, 1185, 1188
  - multiple, 1188
  - multivariate, 1209
  - random number generation, 1018
  - sigma limits, 1183
  - standard variable, 1180
- Gaussian elimination with interchange, 995
- Gaussian integration, 1005–1009
  - points and weights, 1008, 1010
- general tensors
  - algebra, 938–941
  - contraction, 939
  - contravariant, 961
  - covariant, 961
  - dual, 949
  - metric, 957–960
  - physical applications, 957–960, 976
  - pseudotensors, 964
  - tensor densities, 964
- generalised likelihood ratio, 1282
- generating functions
  - associated Laguerre polynomials, 623
  - associated Legendre polynomials, 592
  - Bessel functions, 613
  - Chebyshev polynomials, 601
  - Hermite polynomials, 627
  - Laguerre polynomials, 620
  - Legendre polynomials, 584–586
- generating functions in probability, 1157–1167, *see also* moment generating functions *and* probability generating functions
- geodesics, 797, 976, 982
- geometric distribution, 1159, 1172
- geometric series, 117
- Gibbs' free energy, 178
- Gibbs' phenomenon, 421
- gradient of a function of
  - one variable, 42
  - several real variables, 153–155
- gradient of scalar, 348–352
  - tensor form, 972
- gradient of vector, 937, 969
- gradient operator (grad), 348
  - as integral, 398
  - in curvilinear coordinates, 367
  - in cylindrical polars, 360
  - in spherical polars, 362
  - tensor form, 972
- Gram–Schmidt orthogonalisation of eigenfunctions of Hermitian operators, 562
  - eigenvectors of
    - Hermitian matrices, 277
    - normal matrices, 275
  - functions in a Hilbert space, 557
- gravitational fields and potentials
  - Laplace equation, 679
  - Newton's law, 339
  - Poisson equation, 679, 744
  - uniform disc, 771
  - uniform ring, 742
- Green's functions, 568–571, 751–767
  - and boundary conditions, 512, 514
  - and Dirac  $\delta$ -function, 511
  - and partial differential operators, 753
  - and Wronskian, 527
  - diffusion equation, 749
  - Dirichlet problems, 756–765
    - for ODE, 185, 511–516
  - Neumann problems, 765–767
  - particular integrals from, 514
  - Poisson's equation, 755
- Green's theorems
  - applications, 706, 754, 849
  - in a plane, 384–387, 407
  - in three dimensions, 402
- ground-state energy
  - harmonic oscillator, 796
  - hydrogen atom, 800
- group multiplication tables, 1050
  - order three, 1062
  - order four, 1050, 1052, 1061
  - order five, 1062
  - order six, 1055, 1061
- grouping terms as a test for convergence, 129
- groups
  - Abelian, 1044
  - associative law, 1043
  - cancellation law, 1046
  - centre, 1069
  - closure, 1043
  - cyclic, 1061
  - definition, 1043–1046
  - direct product, 1072
  - division axiom, 1046
  - elements, 1043
    - order, 1047
  - finite, 1043
  - identity element, 1043–1046
  - inverse, 1043, 1046
  - isomorphic, 1051
  - mappings between, 1059–1061
    - homomorphic, 1059–1061
    - image, 1059
    - isomorphic, 1059

## INDEX

- nomenclature, 1102  
 non-Abelian, 1052–1056  
 order, 1043, 1081, 1082, 1094, 1097, 1100  
 permutation law, 1047  
 subgroups, *see* subgroups  
 groups, examples  
 1 and  $-1$  under multiplication, 1043  
 alternating, 1116  
 complex numbers  $e^{i\theta}$ , 1048  
 functions, 1055  
 general linear, 1073  
 integers under addition, 1043  
 integers under multiplication (mod  $N$ ),  
 1049–1051  
 matrices, 1054  
 permutations, 1056–1058  
 quaternion, 1073  
 rotation matrices, 1048  
 symmetries of a square, 1100  
 symmetries of an equilateral triangle, 1047
- $H_n(x)$ , *see* Hermite polynomials  
 Hamilton's principle, 788  
 Hamiltonian, 796  
 Hankel functions  $H_v^{(1)}(x)$ ,  $H_v^{(2)}(x)$ , 607  
 Hankel transforms, 459  
 harmonic oscillators  
 damped, 239, 451  
 ground-state energy, 796  
 Schrödinger equation, 796  
 simple, *see* simple harmonic oscillator  
 heat flow  
 diffusion equation, 678, 696, 723  
 in bar, 723, 749, 770  
 in thin sheet, 698  
 Heaviside function, 441  
 relation to Dirac  $\delta$ -function, 441  
 Heisenberg's uncertainty principle, 435–437  
 Helmholtz equation, 737–741  
 cylindrical polars, 740  
 plane polars, 738  
 spherical polars, 740–741  
 Helmholtz potential, 177  
 hemisphere, centre of mass and centroid, 195  
 Hermite equation, 535, 624–628  
 as example of Sturm–Liouville equation, 566  
 natural interval, 567  
 Hermite polynomials  $H_n(x)$ , 625  
 as special case of confluent hypergeometric  
 function, 634  
 generating function, 627  
 graph of, 625  
 normalisation, 626  
 orthogonality, 626  
 recurrence relations, 628  
 Rodrigues' formula, 626  
 Hermitian conjugate, 256–258  
 and inner product, 258  
 product rule, 257  
 Hermitian forms, 288–292  
 positive definite and semi-definite, 290  
 stationary properties of eigenvectors, 290  
 Hermitian kernel, 816  
 Hermitian matrices, 271  
 eigenvalues, 276–278  
 reality, 276  
 eigenvectors, 276–278  
 orthogonality, 277  
 Hermitian operators, 559–564  
 and physical variables, 650  
 boundary condition for simple harmonic  
 oscillators, 560  
 eigenfunctions  
 completeness, 560, 563  
 orthogonality, 561–563  
 eigenvalues  
 reality, 561  
 Green's functions, 568–571  
 importance of, 555, 560  
 in Sturm–Liouville equations, 564  
 properties, 561–564  
 superposition methods, 568–571  
 higher-order differential equations, *see* ordinary  
 differential equations  
 Hilbert spaces, 557–559  
 hit or miss, in Monte Carlo methods, 1014  
 homogeneous  
 boundary conditions, *see* boundary  
 conditions, homogeneous and  
 inhomogeneous  
 differential equations, 490  
 dimensionally consistent, 475, 521  
 simultaneous linear equations, 293  
 homomorphism, 1059–1061  
 kernel of, 1060  
 representation as, 1083  
 Hooke's law, 953  
 hydrogen atom  
 $s$ -states, 1144  
 electron wavefunction, 208  
 ground-state energy, 800  
 hydrogen molecule, symmetries of, 1041  
 hyperbola  
 as section of quadratic surface, 292  
 equation for, 16  
 hyperbolic functions, 102–109, 833  
 calculus of, 106–109  
 definitions, 102, 833  
 graphs, 102  
 identities, 104  
 in equations, 105  
 inverses, 105  
 graphs, 106  
 trigonometric analogies, 102–104  
 hyperbolic PDE, 687, 690  
 hypergeometric distribution, 1173  
 mean and variance, 1173  
 hypergeometric equation, 535, 628–632  
 as example of Sturm–Liouville equation, 566,  
 567

## INDEX

- general solution, 630  
 natural interval, 567  
 hypergeometric functions, 628–632  
   contiguous relations, 632  
   integral representation, 631  
   recurrence relations, 632  
   special cases, 630  
 hypothesis testing, 1277–1298  
   errors, first and second kind, 1280  
   generalised likelihood ratio, 1282  
   generalised likelihood-ratio test, 1281  
   goodness of fit, 1296  
   Neyman–Pearson test, 1280  
   null, 1278  
   power, 1280  
   rejection region, 1279  
   simple or composite, 1278  
   statistical tests, 1278  
   test statistic, 1278
- i, j, k** (unit vectors), 219  
*i*, square root of  $-1$ , 84  
 identity element of a group, 1043–1046  
   uniqueness, 1043, 1045  
 identity matrices, 254, 255  
 identity operator, 249  
 images, method of, *see* method of images  
 imaginary part or term of a complex number, 83  
 importance sampling, in Monte Carlo methods, 1012  
 improper  
   integrals, 70  
   rotations, 946–948  
 impulses,  $\delta$ -function representation, 441  
 incomplete gamma function, 639  
 independent random variables, 1156, 1200  
 index of a subgroup, 1066  
 indices, of regular singular points, 540  
 indicial equation, 540  
   distinct roots with non-integral difference, 540–542  
   repeated roots, 542, 546, 547  
   roots differ by integer, 542, 546  
 induction, proof by, 31  
 inequalities  
   amongst integrals, 72  
   Bessel, 246, 559  
   Schwarz, 246, 559  
   triangle, 246, 559  
 inertia, *see also* moments of inertia  
   moments and products, 951  
   tensor, 951  
 inexact differentials, 155  
 inexact equation, 473  
 infinite integrals, 70  
   contour integration, 862–867  
 infinite series, *see* series  
 inflection  
   general points of, 52  
   stationary points of, 50–52
- inhomogeneous  
   boundary conditions, *see* boundary conditions, homogeneous and inhomogeneous  
   differential equations, 490  
   simultaneous linear equations, 293  
 inner product in a vector space, *see also* scalar product  
   of finite dimensionality, 244  
   and Hermitian conjugate, 258  
   commutativity, 244  
   distributivity over addition, 244  
   of infinite dimensionality, 557  
 integral equations  
   eigenfunctions, 817  
   eigenvalues, 808, 816  
   Fredholm, 805  
   from differential equations, 803  
   homogeneous, 805  
   linear, 804  
     first kind, 805  
     second kind, 805  
   nomenclature, 804  
   singular, 805  
   Volterra, 805  
 integral equations, methods for  
   differentiation, 812  
   Fredholm theory, 815  
   integral transforms, 809–812  
     Fourier, 809–812  
     Laplace, 810  
   Neumann series, 813–815  
   Schmidt–Hilbert theory, 816–819  
   separable (degenerate) kernels, 807  
 integral functions, 832n  
 integral test for convergence of series, 128  
 integral transforms, *see also* Fourier transforms  
   and Laplace transforms  
   general form, 459  
   Hankel transforms, 459  
   Mellin transforms, 459  
 integrals, *see also* integration  
   complex, *see* complex integrals  
   definite, 59  
   double, *see* multiple integrals  
   Fourier transform of, 444  
   improper, 70  
   indefinite, 62  
   inequalities, 72, 559  
   infinite, 70  
   Laplace transform of, 456  
   limits  
     containing variables, 188  
     fixed, 59  
     variable, 61  
   line, *see* line integrals  
   multiple, *see* multiple integrals  
   non-zero, 1104  
   of vectors, 339  
   properties, 60

## INDEX

- triple, *see* multiple integrals  
 undefined, 59
- integrand, 59
- integrating factor (IF), 506  
 first-order ODE, 473–475
- integration, *see also* integrals  
 applications, 72–76  
   finding the length of a curve, 73  
   mean value of a function, 72  
   surfaces of revolution, 74  
   volumes of revolution, 75  
 as area under a curve, 59  
 as the inverse of differentiation, 61  
 formal definition, 59  
 from first principles, 59  
 in plane polar coordinates, 70  
 logarithmic, 64  
 multiple, *see* multiple integrals  
 multivalued functions, 865–867, 885  
 of Fourier series, 424  
 of functions of several real variables, *see*  
   multiple integrals  
 of hyperbolic functions, 106–109  
 of power series, 135  
 of simple functions, 62  
 of singular functions, 70  
 of sinusoidal functions, 63, 861
- integration constant, 62
- integration, methods for  
 by inspection, 62  
 by parts, 67–69  
 by substitution, 65–67  
    $t$  substitution, 65  
 change of variables, *see* change of variables  
 completing the square, 66  
 contour, *see* contour integration  
 Gaussian, 1005–1009  
 numerical, 1000–1009  
 partial fractions, 64  
 reduction formulae, 69  
 stationary phase, 912–920  
 steepest descents, 908–912  
 trigonometric expansions, 63  
 using complex numbers, 101
- intersection  $\cap$ , probability for, 1120, 1128
- intrinsic derivative, *see* absolute derivative
- invariant tensors, *see* isotropic tensors
- inverse hyperbolic functions, 105
- inverse integral transforms  
 Fourier, 435  
 Laplace, 454, 884–887  
   uniqueness, 454
- inverse matrices, 263–266  
 elements, 264  
 in solution of simultaneous linear equations,  
 295  
 product rule, 266  
 properties, 265
- inverse of a linear operator, 249
- inverse of a product in a group, 1046
- inverse of element in a group  
 uniqueness, 1043, 1046
- inversion theorem, Fourier's, 435
- inversions as  
 improper rotations, 946  
 symmetry operations, 1041
- irregular singular points, 534
- irreps, 1087  
 counting, 1095  
 dimension  $n_i$ , 1097  
 direct sum  $\oplus$ , 1086  
 identity  $A_1$ , 1100, 1104  
 $n$ -dimensional, 1088, 1089, 1102  
 number in a representation, 1087, 1095  
 one-dimensional, 1088, 1089, 1093, 1099, 1102  
 orthogonality theorem, 1090–1092  
 projection operators for, 1107  
 reduction to, 1096  
 summation rules for, 1097–1099
- irrotational vectors, 353
- isobaric ODE, 476  
 non-linear, 521
- isoclines, method of, 1028, 1037
- isomorphic groups, 1051–1056, 1058, 1059
- isomorphism (mapping), 1060
- isotope decay, 484, 525
- isotropic (invariant) tensors, 944–946, 953
- iteration schemes  
 convergence of, 992–994  
 for algebraic equations, 986–994  
 for differential equations, 1025  
 for integral equations, 813–816  
 Gauss–Seidel, 996–998  
 order of convergence, 993
- $J_\nu(x)$ , *see* Bessel functions
- $j$ , square root of  $-1$ , 84
- $j_\nu(x)$ , *see* spherical Bessel functions
- Jacobians  
 analogy with derivatives, 207  
 and change of variables, 206  
 definition in  
   two dimensions, 201  
   three dimensions, 205  
 general properties, 206  
 in terms of a determinant, 201, 205, 207
- joint distributions, *see* bivariate distributions  
*and* multivariate distributions
- Jordan's lemma, 864
- kernel of a homomorphism, 1060, 1063
- kernel of an integral transform, 459
- kernel of integral equations  
 displacement, 809  
 Hermitian, 816  
 of form  $\exp(-ixz)$ , 810–812  
 of linear integral equations, 804  
 resolvent, 814, 815  
 separable (degenerate), 807
- ket vector  $|\psi\rangle$ , 648

## INDEX

- kinetic energy of oscillating system, 316  
 Klein–Gordon equation, 711, 772  
 Kronecker delta  $\delta_{ij}$  and orthogonality, 244  
 Kronecker delta,  $\delta_{ij}$ ,  $\delta_i^j$ , tensor, 928, 941–946,  
   956, 962  
   identities, 943  
   isotropic, 945  
   vector products, 942  
 Kummer function, 633  
 kurtosis, 1150, 1227
- $L_n(x)$ , *see* Laguerre polynomials  
 $L_n^m(x)$ , *see* associated Laguerre polynomials  
 L'Hôpital's rule, 142–144  
 Lagrange equations, 789  
   and energy conservation, 797  
 Lagrange undetermined multipliers, 167–173  
   and ODE eigenvalue estimation, 792  
   application to stationary properties of the  
   eigenvectors of quadratic and Hermitian  
   forms, 290  
   for functions of more than two variables,  
   169–173  
   in deriving the Boltzmann distribution,  
   171–173  
   integral constraints, 785  
   with several constraints, 169–173  
 Lagrange's identity, 226  
 Lagrange's theorem, 1065  
   and the order of a subgroup, 1062  
   and the order of an element, 1062  
 Lagrangian, 789, 797  
 Laguerre equation, 535, 616–621  
   as example of Sturm–Liouville equation, 566,  
   619  
   natural interval, 567, 619  
 Laguerre polynomials  $L_n(x)$ , 617  
   as special case of confluent hypergeometric  
   function, 634  
   generating function, 620  
   graph of, 617  
   normalisation, 619  
   orthogonality, 619  
   recurrence relations, 620  
   Rodrigues' formula, 618  
 Lamé constants, 953  
 lamina: mass, centre of mass and centroid,  
   193–195  
 Laplace equation, 679  
   expansion methods, 741–744  
   in two dimensions, 688, 690, 717, 718  
   and analytic functions, 829  
   and conformal transformations, 876–879  
   numerical method for, 1031, 1038  
   plane polars, 725–727  
   separated variables, 717  
   in three dimensions  
   cylindrical polars, 728–731  
   spherical polars, 731–737  
   uniqueness of solution, 741  
   with specified boundary values, 764, 766  
 Laplace expansion, 259  
 Laplace transforms, 453–459, 884  
   convolution  
     associativity, commutativity, distributivity,  
     458  
     definition, 457  
   convolution theorem, 457  
   definition, 453  
   for ODE with constant coefficients, 501–503  
   for PDE, 747–748  
   inverse, 454, 884–887  
   uniqueness, 454  
   properties: translation, exponential  
   multiplication, etc., 456  
   table for common functions, 455  
 Laplace transforms, examples  
   constant, 453  
   derivatives, 455  
   exponential function, 453  
   integrals, 456  
   polynomial, 453  
 Laplacian, *see* del squared  $\nabla^2$  (Laplacian)  
 Laurent expansion, 855–858  
   analytic and principal parts, 855  
   region of convergence, 855  
 least squares, method of, 1271–1277  
   basis functions, 1273  
   linear, 1272  
   non-linear, 1276  
   response matrix, 1273  
 Legendre equation, 534, 535, 577–586  
   as example of Sturm–Liouville equation, 566,  
   583  
   associated, *see* associated Legendre equation  
   general solution, 578, 580  
   natural interval, 567, 583  
 Legendre functions  $P_\ell(x)$ , 577–586  
   associated Legendre functions, 768  
   of second kind  $Q_\ell(x)$ , 579  
   graph of, 580  
 Legendre linear equation, 503  
 Legendre polynomials  $P_\ell(x)$ , 578  
   as special case of hypergeometric function,  
   631  
   associated Legendre functions, 733  
   generating function, 584–586  
   graph of, 579  
   in Gaussian integration, 1006  
   normalisation, 578, 582  
   orthogonality, 583, 735  
   recurrence relations, 585, 586  
   Rodrigues' formula, 581  
 Leibnitz' rule for differentiation of integrals, 178  
 Leibnitz' theorem, 48–50  
 length of  
   a vector, 218  
   plane curves, 73, 341  
   space curves, 341  
   tensor form, 982

## INDEX

- level lines, 905, 906  
 Levi-Civita symbol, *see*  $\epsilon_{ijk}$ , Levi-Civita symbol, tensor  
 likelihood function, 1255  
 limits, 141–144  
   definition, 141  
   L'Hôpital's rule, 142–144  
   of functions containing exponents, 142  
   of integrals, 59  
     containing variables, 188  
   of products, 141  
   of quotients, 141–144  
   of sums, 141  
 line charge, electrostatic potential, 872, 878  
 line integrals  
   and Cauchy integrals, 851–853  
   and Stokes' theorem, 406–409  
   of scalars, 377–387  
   of vectors, 377–389  
   physical examples, 381  
   round closed loop, 386  
 line of steepest descents, 908  
 line, vector equation of, 226  
 linear dependence and independence  
   definition in a vector space, 242  
   of basis vectors, 217  
   relationship with rank, 267  
 linear differential operator  $\mathcal{L}$ , 511, 545, 554  
   adjoint  $\mathcal{L}^\dagger$ , 559  
   eigenfunctions, *see* eigenfunctions  
   eigenvalues, *see* eigenvalues, of linear differential operators  
   for Sturm-Liouville equation, 564–568  
   Hermitian, 555, 559–564  
   self-adjoint, 559  
 linear equations, differential  
   first-order ODE, 474  
   general ODE, 490–517  
   ODE with constant coefficients, 492–503  
   ODE with variable coefficients, 503–517  
 linear equations, simultaneous, *see* simultaneous linear equations  
 linear independence of functions, 491  
   Wronskian test, 491, 532  
 linear integral operator  $\mathcal{K}$ , 805  
   and Schmidt–Hilbert theory, 816–818  
   Hermitian conjugate, 805  
   inverse, 806  
 linear interpolation for algebraic equations, 988  
 linear least squares, method of, 1272  
 linear molecules  
   normal modes of, 320–322  
   symmetries of, 1077  
 linear operators, 247–249  
   associativity, 249  
   distributivity over addition, 249  
   eigenvalues and eigenvectors, 272  
   in a particular basis, 248  
   inverse, 249  
   non-commutativity, 249  
   particular: identity, null or zero, singular and non-singular, 249  
   properties, 249  
 linear vector spaces, *see* vector spaces  
 lines of steepest descent, 906  
 Liouville's theorem, 853  
 Ln of a complex number, 99, 834  
 ln (natural logarithm)  
   Maclaurin series for, 140  
   of a complex number, 99, 834  
 log-likelihood function, 1258  
 longitudinal vibrations in a rod, 677  
 lottery (UK), and hypergeometric distribution, 1174  
 lower triangular matrices, 269  
 Maclaurin series, 138  
   standard expressions, 140  
 Madelung constant, 149  
 magnetic dipole, 220  
 magnitude of a vector, 218  
   in terms of scalar or dot product, 221  
 mappings between groups, *see* groups, mappings between  
 marginal distributions, 1198  
 mass of non-uniform bodies, 193  
 matrices, 241–307  
   as a vector space, 252  
   as arrays of numbers, 249  
   as representation of a linear operator, 249  
   column, 250  
   elements, 249  
   minors and cofactors, 259  
   identity or unit, 254  
   row, 250  
   zero or null, 254  
 matrices, algebra of, 250  
   addition, 251  
   change of basis, 283–285  
   Cholesky separation, 313  
 diagonalisation, *see* diagonalisation of matrices  
 multiplication, 252–254  
   and common eigenvalues, 278  
   commutator, 309  
   non-commutativity, 254  
 multiplication by a scalar, 251  
 normal modes, *see* normal modes  
 numerical methods, *see* numerical methods for simultaneous linear equations  
 similarity transformations, *see* similarity transformations  
 simultaneous linear equations, *see* simultaneous linear equations  
 subtraction, 251  
 matrices, derived  
   adjoint, 256–258  
   complex conjugate, 256–258  
   Hermitian conjugate, 256–258  
   inverse, *see* inverse matrices

## INDEX

- transpose, 250
- matrices, properties of
- anti- or skew-symmetric, 270
  - anti-Hermitian, *see* anti-Hermitian matrices
  - determinant, *see* determinants
  - diagonal, 268
  - eigenvalues, *see* eigenvalues
  - eigenvectors, *see* eigenvectors
  - Hermitian, *see* Hermitian matrices
  - normal, *see* normal matrices
  - nullity, 293
  - order, 249
  - orthogonal, 270
  - rank, 267
  - square, 249
  - symmetric, 270
  - trace or spur, 258
  - triangular, 269
  - tridiagonal, 998–1000, 1030
  - unitary, *see* unitary matrices
- matrix elements in quantum mechanics
- as integrals, 1103
  - dipole, 1108, 1115
- maxima and minima (local) of a function of constrained variables, *see* Lagrange undetermined multipliers
- one real variable, 50–52
    - sufficient conditions, 51
  - several real variables, 162–167
    - sufficient conditions, 164, 167
- maximum modulus theorem, 881
- maximum-likelihood, method of, 1255–1271
- and Bayesian approach, 1264
  - bias, 1260
  - data modelling, 1255
  - estimator, 1256
  - extended, 1270
  - log-likelihood function, 1258
  - parameter estimation, 1255
  - transformation invariance, 1260
- Maxwell's
- electromagnetic equations, 373, 408, 979
  - thermodynamic relations, 176–178
- Maxwell–Boltzmann statistics, 1138
- mean  $\mu$
- from MGF, 1163
  - from PGF, 1158
  - of RVD, 1144
  - of sample, 1223
  - of sample: geometric, harmonic, root mean square, 1223
- mean value of a function of
- one variable, 72
  - several variables, 199
- mean value theorem, 56
- median of RVD, 1145
- membrane
- deformed rim, 725–727
  - normal modes, 739, 1112
  - transverse vibrations, 677, 739, 768, 1112
- method of images, 706, 758–765, 878
    - disc (section of cylinder), 764, 766
    - infinite plate, 759
    - intersecting plates in two dimensions, 761
    - sphere, 762–764, 772
  - metric tensor, 957–960, 963
    - and Christoffel symbols, 966
    - and scale factors, 957, 972
    - covariant derivative of, 982
    - determinant, 957, 964
      - derivative of, 973
    - length element, 957
    - raising or lowering index, 959, 963
    - scalar product, 958
    - volume element, 957, 981
  - MGF, *see* moment generating functions
  - Milne's method, 1022
  - minimum-variance estimator, 1232
  - minor of a matrix element, 259
  - mixed, components of tensor, 957, 962, 969
  - ML estimators, 1256
    - bias, 1260
    - confidence limits, 1262
    - efficiency, 1261
    - transformation invariance, 1260
  - mode of RVD, 1145
  - modulo, mod  $N$ , multiplication, 1049
  - modulus
    - of a complex number, 87
    - of a vector, *see* magnitude of a vector
  - molecules
    - bonding in, 1103, 1105–1108
    - dipole moments of, 1077
    - symmetries of, 1077
  - moment generating functions (MGFs), 1162–1167
    - and central limit theorem, 1195
    - and PGF, 1163
    - mean and variance, 1163
    - particular distributions
      - binomial, 1170
      - exponential, 1191
      - Gaussian, 1163, 1185
      - Poisson, 1177
    - properties, 1163
  - moments (of distributions)
    - central, 1148
    - of RVD, 1147
  - moments (of forces), vector representation of, 223
  - moments of inertia
    - and inertia tensor, 951
    - definition, 198
    - of disc, 208
    - of rectangular lamina, 198
    - of right circular cylinder, 209
    - of sphere, 205
    - perpendicular axes theorem, 209
    - momentum as first-order tensor, 933
  - monomorphism, 1061

## INDEX

- Monte Carlo methods, of integration, 1009–1017  
 antithetic variates, 1014  
 control variates, 1013  
 crude, 1011  
 hit or miss, 1014  
 importance sampling, 1012  
 multiple integrals, 1016  
 random number generation, 1017  
 stratified sampling, 1012
- Morera's theorem, 851
- multinomial distribution, 1208  
 and multiple Poisson distribution, 1218
- multiple angles, trigonometric formulae, 10
- multiple integrals  
 application in finding  
 area and volume, 191–193  
 mass, centre of mass and centroid, 193–195  
 mean value of a function of several variables, 199  
 moments of inertia, 198  
 change of variables  
 double integrals, 200–204  
 general properties, 206  
 triple integrals, 204  
 definitions of  
 double integrals, 187  
 triple integrals, 190  
 evaluation, 188–190  
 notation, 188, 189, 191  
 order of integration, 188, 191
- multiplication tables for groups, *see* group multiplication tables
- multiplication theorem, *see* Parseval's theorem
- multivalued functions, 835–837  
 integration of, 865–867
- multivariate distributions, 1196, 1207–1211  
 change of variables, 1206  
 Gaussian, 1209  
 multinomial, 1208
- mutually exclusive events, 1120, 1129
- $n_r(x)$ , *see* spherical Bessel functions
- nabla  $\nabla$ , *see* gradient operator (grad)
- natural interval  
 for associated Laguerre equation, 567, 622  
 for associated Legendre equation, 567, 590, 591  
 for Bessel equation, 608  
 for Chebyshev equation, 567, 599  
 for Hermite equation, 567  
 for Laguerre equation, 567, 619  
 for Legendre equation, 567, 583  
 for simple harmonic oscillator equation, 567  
 for Sturm–Liouville equations, 565, 567
- natural logarithm, *see*  $\ln$  and  $\ln$
- natural numbers, in series, 31, 121
- natural representations, 1081, 1110
- necessary and sufficient conditions, 34
- negative binomial distribution, 1172
- negative function, 556
- negative vector, 242
- Neumann boundary conditions, 702  
 Green's functions, 754, 765–767  
 method of images, 765–767  
 self-consistency, 765
- Neumann functions  $Y_\nu(x)$ , 607
- Neumann series, 813–815
- Newton–Raphson (NR) method, 990–992  
 order of convergence, 993
- Neyman–Pearson test, 1280
- nodes of oscillation, 693
- non-Abelian groups, 1052–1056  
 of functions, 1055  
 of matrices, 1054  
 of permutations, 1056–1058  
 of rotations and reflections, 1052
- non-Cartesian coordinates, *see* curvilinear, cylindrical polar, plane polar and spherical polar coordinates
- non-linear differential equations, *see* ordinary differential equations, non-linear
- non-linear least squares, method of, 1276
- norm of  
 function, 557  
 vector, 244
- normal  
 to coordinate surface, 366  
 to plane, 228  
 to surface, 346, 350, 390
- normal derivative, 350
- normal distribution, *see* Gaussian (normal) distribution
- normal matrices, 272  
 eigenvectors  
 completeness, 275  
 orthogonality, 275  
 eigenvectors and eigenvalues, 273–276
- normal modes, 316–329  
 characteristic equation, 319  
 coupled pendulums, 329, 331  
 definition, 320  
 degeneracy, 1110–1113  
 frequencies of, 319  
 linear molecular system, 320–322  
 membrane, 739, 1112  
 normal coordinates, 320  
 normal equations, 320  
 rod–string system, 317–320  
 symmetries of, 322
- normal subgroups, 1063
- normalisation of  
 eigenfunctions, 562  
 eigenvectors, 273  
 functions, 557  
 vectors, 219
- null (zero)  
 matrix, 254, 255  
 operator, 249  
 space, of a matrix, 293  
 vector, 214, 242, 556

## INDEX

- null operation, as identity element of group, 1044  
 nullity, of a matrix, 293  
 numerical methods for algebraic equations, 985–992  
   binary chopping, 990  
   convergence of iteration schemes, 992–994  
   linear interpolation, 988  
   Newton–Raphson, 990–992  
   rearrangement methods, 987  
 numerical methods for integration, 1000–1009  
   Gaussian integration, 1005–1009  
   mid-point rule, 1034  
   Monte Carlo, 1009  
   nomenclature, 1001  
   Simpson’s rule, 1004  
   trapezium rule, 1002–1004  
 numerical methods for ordinary differential equations, 1020–1030  
   accuracy and convergence, 1021  
   Adams method, 1024  
   difference schemes, 1021–1023  
   Euler method, 1021  
   first-order equations, 1021–1028  
   higher-order equations, 1028–1030  
   isoclines, 1028  
   Milne’s method, 1022  
   prediction and correction, 1024–1026  
   reduction to matrix form, 1030  
   Runge–Kutta methods, 1026–1028  
   Taylor series methods, 1023  
 numerical methods for partial differential equations, 1030–1032  
   diffusion equation, 1032  
   Laplace’s equation, 1031  
   minimising error, 1032  
 numerical methods for simultaneous linear equations, 994–1000  
   Gauss–Seidel iteration, 996–998  
   Gaussian elimination with interchange, 995  
   matrix form, 994–1000  
   tridiagonal matrices, 998–1000  
  
 $O(x)$ , order of, 132  
 observables in quantum mechanics, 277, 560  
 odd functions, *see* antisymmetric functions  
 ODE, *see* ordinary differential equations (ODEs)  
 operators  
   Hermitian, *see* Hermitian operators  
   linear, *see* linear operators *and* linear differential operator *and* linear integral operator  
 operators (quantum)  
   angular momentum, 656–663  
   annihilation and creation, 667  
   coordinate-free, 648–671  
   eigenvalues and eigenstates, 649  
   physical examples  
     angular momentum, 658  
     Hamiltonian, 657  
  
 order of  
   approximation in Taylor series, 137n  
   convergence of iteration schemes, 993  
   group, 1043  
   group element, 1047  
   ODE, 468  
   permutation, 1058  
   recurrence relations (series), 497  
   subgroup, 1061  
     and Lagrange’s theorem, 1065  
   tensor, 930  
 ordinary differential equations (ODE), *see also*  
   differential equations, particular  
   boundary conditions, 468, 470, 501  
   complementary function, 491  
   degree, 468  
   dimensionally homogeneous, 475  
   exact, 472, 505  
   first-order, 468–484  
   first-order higher-degree, 480–484  
     soluble for  $p$ , 480  
     soluble for  $x$ , 481  
     soluble for  $y$ , 482  
   general form of solution, 468–470  
   higher-order, 490–523  
   homogeneous, 490  
   inexact, 473  
   isobaric, 476, 521  
   linear, 474, 490–517  
   non-linear, 518–523  
     exact, 519  
     isobaric (homogeneous), 521  
      $x$  absent, 518  
      $y$  absent, 518  
   order, 468  
   ordinary point, *see* ordinary points of ODE  
   particular integral (solution), 469, 492, 494  
   singular point, *see* singular points of ODE  
   singular solution, 469, 481, 482, 484  
 ordinary differential equations, methods for  
   canonical form for second-order equations, 516  
   eigenfunctions, 554–573  
   equations containing linear forms, 478–480  
   equations with constant coefficients, 492–503  
   Green’s functions, 511–516  
   integrating factors, 473–475  
   Laplace transforms, 501–503  
   numerical, 1020–1030  
   partially known CF, 506  
   separable variables, 471  
   series solutions, 531–550, 604  
   undetermined coefficients, 494  
   variation of parameters, 508–510  
 ordinary points of ODE, 533, 535–538  
   indicial equation, 543  
 orthogonal lines, condition for, 12  
 orthogonal matrices, 270, 929, 930  
   general properties, *see* unitary matrices  
   orthogonal systems of coordinates, 364

## INDEX

- orthogonal transformations, 932
- orthogonalisation (Gram–Schmidt) of
- eigenfunctions of an Hermitian operator, 562
  - eigenvectors of a normal matrix, 275
  - functions in a Hilbert space, 557
- orthogonality of
- eigenfunctions of an Hermitian operator, 561–563
  - eigenvectors of a normal matrix, 275
  - eigenvectors of an Hermitian matrix, 277
  - functions, 557
  - terms in Fourier series, 417, 425
  - vectors, 219, 244
- orthogonality properties of characters, 1094, 1102
- orthogonality theorem for irreps, 1090–1092
- orthonormal
- basis functions, 557
  - basis vectors, 244
  - under unitary transformation, 285
- oscillations, *see* normal modes
- outcome, of trial, 1119
- outer product of two vectors, 936
- $P_\nu(x)$ , *see* Legendre polynomials
- $P_\nu^m(x)$ , *see* associated Legendre functions
- Pappus' theorems, 195–197
- parabola, equation for, 16
- parabolic PDE, 687, 690
- parallel axis theorem, 238
- parallel vectors, 223
- parallelepiped, volume of, 225
- parallelogram equality, 247
- parallelogram, area of, 223, 224
- parameter estimation (statistics), 1229–1255, 1298
- Bessel correction, 1248
  - error in mean, 1298
  - maximum-likelihood, 1255
  - mean, 1243
  - variance, 1245–1248
- parameters, variation of, 508–510
- parametric equations
- of conic sections, 17
  - of cycloid, 370, 785
  - of space curves, 340
  - of surfaces, 345
- parity inversion, 1102
- Parseval's theorem
- conservation of energy, 451
  - for Fourier series, 426
  - for Fourier transforms, 450
- partial derivative, *see* partial differentiation
- partial differential equations (PDE), 675–707, 713–767, *see also* differential equations, particular
- arbitrary functions, 680–685
  - boundary conditions, 681, 699–707, 723
  - characteristics, 699–705
  - and equation type, 703
  - equation types, 687, 710
  - first-order, 681–687
  - general solution, 681–692
  - homogeneous, 685
  - inhomogeneous equation and problem, 685–687, 744–746, 751–767
  - particular solutions (integrals), 685–692
  - second-order, 687–698
- partial differential equations (PDE), methods for
- change of variables, 691, 696–698
  - constant coefficients, 687
  - general solution, 689
  - integral transform methods, 747–751
  - method of images, *see* method of images
  - numerical, 1030–1032
  - separation of variables, *see* separation of variables
  - superposition methods, 717–724
  - with no undifferentiated term, 684
- partial differentiation, 151–179
- as gradient of a function of several real variables, 151
  - chain rule, 157
  - change of variables, 158–160
  - definitions, 151–153
  - properties, 157
  - cyclic relation, 157
  - reciprocity relation, 157
- partial fractions, 18–25
- and degree of numerator, 21
  - as a means of integration, 64
  - complex roots, 22
  - in inverse Laplace transforms, 454, 502
  - repeated roots, 23
- partial sum, 115
- particular integrals (PI), 469, *see also* ordinary differential equation, methods for *and* partial differential equations, methods for
- partition of a
- group, 1064
  - set, 1065
- parts, integration by, 67–69
- path integrals, *see* line integrals
- PDE, *see* partial differential equations
- PDFs, 1140
- pendulums, coupled, 329, 331
- periodic function representation, *see* Fourier series
- permutation groups  $S_n$ , 1056–1058
- cycle notation, 1057
- permutation law in a group, 1047
- permutations, 1133–1139
- degree, 1056
  - distinguishable, 1135
  - order of, 1058
  - symbol  ${}^n P_k$ , 1133
- perpendicular axes theorem, 209
- perpendicular vectors, 219, 244
- PF, *see* probability functions
- PGF, *see* probability generating functions

## INDEX

- phase memory, 895  
 phase, complex, 896  
 PI, *see* particular integrals  
 plane curves, length of, 73  
   in Cartesian coordinates, 73  
   in plane polar coordinates, 74  
 plane polar coordinates, 70, 336  
   arc length, 74, 361  
   area element, 202, 361  
   basis vectors, 336  
   velocity and acceleration, 337  
 plane waves, 695, 716  
 planes  
   and simultaneous linear equations, 300  
   vector equation of, 227  
 plates, conducting, *see also* complex potentials,  
   for plates  
   line charge near, 761  
   point charge near, 759  
 point charges,  $\delta$ -function representation, 441  
 point groups, 1082  
 points of inflection of a function of  
   one real variable, 50–52  
   several real variables, 162–167  
 Poisson distribution  $Po(\lambda)$ , 1174–1179  
   and Gaussian distribution, 1187  
   as limit of binomial distribution, 1174, 1177  
   mean and variance, 1176  
   MGF, 1177  
   multiple, 1178  
   recurrence formula, 1176  
 Poisson equation, 575, 679, 744–746  
   fundamental solution, 757  
   Green's functions, 753–767  
   uniqueness, 705–707  
 Poisson summation formula, 461  
 Poisson's ratio, 953  
 polar coordinates, *see* plane polar *and* cylindrical  
   polar *and* spherical polar coordinates  
 polar representation of complex numbers, 92–95  
 polar vectors, 949  
 pole, of a function of a complex variable  
   contours containing, 861–867, 884–887  
   order, 837, 856  
   residue, 856–858  
 polynomial equations, 1–10  
   conjugate roots, 99  
   factorisation, 7  
   multiplicities of roots, 4  
   number of roots, 83, 85, 868  
   properties of roots, 9  
   real roots, 1  
   solution of, using de Moivre's theorem, 98  
 polynomial solutions of ODE, 538, 548–550  
 populations, sampling of, 1222  
 positive (semi-) definite quadratic and Hermitian  
   forms, 290  
 positive semi-definite norm, 244  
 potential energy of  
   ion in a crystal lattice, 148  
   magnetic dipole in a field, 220  
   oscillating system, 317  
 potential function  
   and conservative fields, 389  
   complex, 871–876  
   electrostatic, *see* electrostatic fields and  
   potentials  
   gravitational, *see* gravitational fields and  
   potentials  
   vector, 389  
 power series  
   and differential equations, *see* series solutions  
   of differential equations  
   interval of convergence, 132  
   Maclaurin, *see* Maclaurin series  
   manipulation: difference, differentiation,  
   integration, product, substitution, sum, 134  
   Taylor, *see* Taylor series  
 power series in a complex variable, 133, 830–832  
   analyticity, 832  
   circle and radius of convergence, 133, 831  
   convergence tests, 831, 832  
   form, 830  
   power, in hypothesis testing, 1280  
   powers, complex, 99, 833  
 prediction and correction methods, 1024–1026,  
   1035  
 prime, non-existence of largest, 34  
 principal axes of  
   Cartesian tensors, 951–953  
   conductivity tensors, 952  
   inertia tensors, 951  
   quadratic surfaces, 292  
   rotation symmetry, 1102  
 principal normals of space curves, 342  
 principal value of  
   complex integrals, 864  
   complex logarithms, 100, 834  
   principle of the argument, 880  
 probability, 1124–1211  
   axioms, 1125  
   conditional, 1128–1133  
   Bayes' theorem, 1132  
   combining, 1130  
   definition, 1125  
   for intersection  $\cap$ , 1120  
   for union  $\cup$ , 1121, 1125–1128  
 probability distributions, 1139, *see also*  
   *individual distributions*  
   bivariate, *see* bivariate distributions  
   change of variables, 1150–1157  
   cumulants, 1166  
   generating functions, *see* moment generating  
   functions *and* probability generating  
   functions  
   mean  $\mu$ , 1144  
   mean of functions, 1145  
   mode, median and quartiles, 1145  
   moments, 1147–1150  
   multivariate, *see* multivariate distributions

## INDEX

- standard deviation  $\sigma$ , 1146  
 variance  $\sigma^2$ , 1146  
 probability functions (PFs), 1139  
   cumulative (CPFs), 1139, 1141  
   density functions (PDFs), 1140  
 probability generating functions (PGFs),  
   1157–1162  
   and MGF, 1163  
   binomial, 1161  
   definition, 1158  
   geometric, 1159  
   mean and variance, 1158  
   Poisson, 1158  
   sums of RV, 1161  
   trials, 1158  
   variable sums of RV, 1161  
 product rule for differentiation, 44–46, 48–50  
 products of inertia, 951  
 projection operators for irreps, 1107, 1116  
 projection tensors, 979  
 proper rotations, 946  
 proper subgroups, 1061  
 pseudoscalars, 947, 950  
 pseudotensors, 946–950, 964  
 pseudovectors, 946–950  
  
 $Q_\nu(x)$ , *see* Legendre polynomials  
 $Q_\nu^m(x)$ , *see* associated Legendre functions  
 quadratic equations  
   complex roots of, 83  
   properties of roots, 10  
   roots of, 2  
 quadratic forms, 288–292  
   completing the square, 1206  
   positive definite and semi-definite, 290  
   quadratic surfaces, 292  
   removing cross terms, 289  
   stationary properties of eigenvectors, 290  
 quantum mechanics, from classical mechanics,  
   657  
 quantum operators, *see* operators (quantum)  
 quartiles, of RVD, 1145  
 quaternion group, 1073, 1113  
 quotient law for tensors, 939–941  
 quotient rule for differentiation, 47  
 quotient test for series, 127  
  
 radius of convergence, 133, 831  
 radius of curvature  
   of plane curves, 53  
   of space curves, 342  
 radius of torsion of space curves, 343  
 random number generation, 1017  
 random numbers, non-uniform distribution,  
   1035  
 random variable distributions, *see* probability  
   distributions  
 random variables (RV), 1119, 1139–1143  
   continuous, 1140–1143  
   dependent, 1196–1205  
   discrete, 1139  
   independent, 1156, 1200  
   sums of, 1160–1162  
   uncorrelated, 1200  
 range of a matrix, 293  
 rank of matrices, 267  
   and determinants, 267  
   and linear dependence, 267  
 rank of tensors, *see* order of tensor  
 rate of change of a function of  
   one real variable, 41  
   several real variables, 153–155  
 ratio comparison test, 127  
 ratio test (D'Alembert), 126, 832  
   in convergence of power series, 132  
 ratio theorem, 215  
   and centroid of a triangle, 216  
 Rayleigh–Ritz method, 327–329, 800  
 real part or term of a complex number, 83  
 real roots, of a polynomial equation, 1  
 rearrangement methods for algebraic equations,  
   987  
 reciprocal vectors, 233, 366, 955, 959  
 reciprocity relation for partial derivatives, 157  
 rectangular distribution, 1194  
   Fourier transform of, 442  
 recurrence relations (functions), 585, 611  
   associated Laguerre polynomials, 624  
   associated Legendre functions, 592  
   Chebyshev polynomials, 601  
   confluent hypergeometric functions, 635  
   Hermite polynomials, 628  
   hypergeometric functions, 632  
   Laguerre polynomials, 620  
   Legendre polynomials, 586  
 recurrence relations (series), 496–501  
   characteristic equation, 499  
   coefficients, 536, 538, 999  
   first-order, 497  
   second-order, 499  
   higher-order, 501  
 reducible representations, 1084, 1086  
 reduction formulae for integrals, 69  
 reflections  
   and improper rotations, 946  
   as symmetry operations, 1041  
 reflexivity, and equivalence relations, 1064  
 regular functions, *see* analytic functions  
 regular representations, 1097, 1110  
 regular singular points, 534, 538–540  
 relative velocities, 218  
 remainder term in Taylor series, 138  
 repeated roots of auxiliary equation, 493  
 representation, 1076  
   definition, 1082  
   dimension of, 1078, 1082  
   equivalent, 1084–1086  
   faithful, 1083, 1098  
   generation of, 1078–1084, 1112  
   irreducible, *see* irreps

## INDEX

- natural, 1081, 1110  
 product, 1103–1105  
 reducible, 1084, 1086  
 regular, 1097, 1110  
   counting irreps, 1098  
 unitary, 1086
- representative matrices, 1079  
 block-diagonal, 1086  
 eigenvalues, 1100  
 inverse, 1083  
 number needed, and order of group, 1082  
 of identity, 1082
- residue  
   at a pole, 856–858  
   theorem, 858–860
- resolution function, 446  
 resolvent kernel, 814, 815  
 response matrix, for linear least squares, 1273  
 rhomboid, volume of, 237  
 Riemann tensor, 981  
 Riemann theorem for conditional convergence, 124  
 Riemann zeta series, 128, 129  
 right hand screw rule, 222  
 Rodrigues' formula for  
   associated Laguerre polynomials, 622  
   associated Legendre functions, 588  
   Chebyshev polynomials, 599  
   Hermite polynomials, 626  
   Laguerre polynomials, 618  
   Legendre polynomials, 581
- Rolle's theorem, 55  
 root test (Cauchy), 129, 831  
 roots  
   of a polynomial equation, 2  
   properties, 9  
   of unity, 97  
 rope, suspended at its ends, 786  
 rotation groups (continuous), invariant subspaces, 1088  
 rotation matrices as a group, 1048  
 rotation of a vector, *see* curl  
 rotations  
   as symmetry operations, 1041  
   axes and orthogonal matrices, 930, 931, 961  
   improper, 946–948  
   invariance under, 934  
   product of, 931  
   proper, 946
- Rouché's theorem, 880–882  
 row matrix, 250  
 Runge–Kutta methods, 1026–1028  
 RV, *see* random variables  
 RVD (random variable distributions), *see* probability distributions
- saddle point method of integration, 908  
 saddle points, 162  
   and integral evaluation, 905  
   sufficient conditions, 164, 167
- sampling  
   correlation, 1227  
   covariance, 1227  
   space, 1119  
   statistics, 1222–1229  
   with or without replacement, 1129
- scalar fields, 347  
 derivative along a space curve, 349  
 gradient, 348–352  
 line integrals, 377–387  
 rate of change, 349
- scalar product, 219–222  
 and inner product, 244  
 and metric tensor, 958  
 and perpendicular vectors, 219, 244  
 for vectors with complex components, 221  
 in Cartesian coordinates, 221  
 invariance, 930, 939
- scalar triple product, 224–226  
 cyclic permutation of, 225  
 in Cartesian coordinates, 225  
 determinant form, 225  
 interchange of dot and cross, 225
- scalars, 212  
 invariance, 930  
 zero-order tensors, 933
- scale factors, 359, 362, 364  
 and metric tensor, 957, 972
- scattering in quantum mechanics, 463  
 Schmidt–Hilbert theory, 816–819  
 Schrödinger equation, 679  
 constant potential, 768  
 hydrogen atom, 741  
 numerical solution, 1039  
 variational approach, 795
- Schwarz inequality, 246, 559  
 Schwarz–Christoffel transformation, 843  
 second differences, 1019  
 second-order differential equations, *see* ordinary differential equations *and* partial differential equations
- secular determinant, 280  
 self-adjoint operators, 559–564, *see also* Hermitian operators
- semicircle, angle in, 18  
 semicircular lamina, centre of mass, 197  
 separable kernel in integral equations, 807  
 separable variables in ODE, 471  
 separation constants, 715, 717  
 separation of variables, for PDE, 713–746  
 diffusion equation, 716, 722–724, 737, 751  
 expansion methods, 741–744  
 general method, 713–717  
 Helmholtz equation, 737–741  
 inhomogeneous boundary conditions, 722–724  
 inhomogeneous equations, 744–746  
 Laplace equation, 717–722, 725–737, 741  
 polar coordinates, 725–746  
 separation constants, 715, 717  
 superposition methods, 717–724

## INDEX

- wave equation, 714–716, 737, 739
- series, 115–141
- convergence of, *see* convergence of infinite series
  - differentiation of, 131
  - finite and infinite, 116
  - integration of, 131
  - multiplication by a scalar, 131
  - multiplication of (Cauchy product), 131
  - notation, 116
  - operations, 131
  - summation, *see* summation of series
- series, particular
- arithmetic, 117
  - arithmetic-geometric, 118
  - Fourier, *see* Fourier series
  - geometric, 117
  - Maclaurin, 138, 140
  - power, *see* power series
  - powers of natural numbers, 121
  - Riemann zeta, 128, 129
  - Taylor, *see* Taylor series
- series solutions of differential equations, 531–550
- about ordinary points, 535–538
  - about regular singular points, 538–540
    - Frobenius series, 539
  - convergence, 536
  - indicial equation, 540
  - linear independence, 540
  - polynomial solutions, 538, 548–550
  - recurrence relation, 536, 538
  - second solution, 537, 544–548
    - derivative method, 545–548
    - Wronskian method, 544, 580
- shortest path, 778
- and geodesics, 976, 982
- similarity transformations, 283–285, 929, 1092
- properties of matrix under, 284
  - unitary transformations, 285
- simple harmonic oscillator, 555
- energy levels of, 642, 902
  - equation, 535, 566
  - operator formalism, 667
- simple poles, 838
- Simpson's rule, 1004
- simultaneous linear equations, 292–307
- and intersection of planes, 300
  - homogeneous and inhomogeneous, 293
  - singular value decomposition, 301–307
  - solution using
    - Cramer's rule, 299
    - inverse matrix, 295
    - numerical methods, *see* numerical methods for simultaneous linear equations
- sine,  $\sin(x)$
- in terms of exponential functions, 102
  - Maclaurin series for, 140
  - orthogonality relations, 417
- singular and non-singular
- integral equations, 805
  - linear operators, 249
  - matrices, 263
- singular integrals, *see* improper integrals
- singular point (singularity), 826, 837–839
- essential, 838, 856
  - removable, 838
- singular points of ODE, 533
- irregular, 534
  - particular equations, 535
  - regular, 534
- singular solution of ODE, 469, 481, 482, 484
- singular value decomposition
- and simultaneous linear equations, 301–307
  - singular values, 302
- $\sinh$ , hyperbolic sine, 102, 833, *see also*
- hyperbolic functions
- skew-symmetric matrices, 270
- skewness, 1150, 1227
- Snell's law, 788
- soap films, 780
- solenoidal vectors, 352, 389
- solid angle
- as surface integral, 395
  - subtended by rectangle, 411
- solid: mass, centre of mass and centroid, 193–195
- source density, 679
- space curves, 340–344
- arc length, 341
  - binormal, 342
  - curvature, 342
  - Frenet–Serret formulae, 343
  - parametric equations, 340
  - principal normal, 342
  - radius of curvature, 342
  - radius of torsion, 343
  - tangent vector, 342
  - torsion, 342
- spaces, *see* vector spaces
- span of a set of vectors, 242
- sphere, vector equation of, 228
- spherical Bessel functions, 615, 741
- of first kind  $j_\nu(z)$ , 615
  - of second kind  $n_\nu(z)$ , 615
- spherical harmonics  $Y_\nu^m(\theta, \phi)$ , 593–595
- addition theorem, 594
- spherical polar coordinates, 361–363
- area element, 362
  - basis vectors, 362
  - length element, 362
  - vector operators, 361–363
  - volume element, 205, 362
- spur of a matrix, *see* trace of a matrix
- square matrices, 249
- square, symmetries of, 1100
- square-wave, Fourier series for, 418
- stagnation points of fluid flow, 873
- standard deviation  $\sigma$ , 1146
- of sample, 1224
- standing waves, 693

## INDEX

- stationary phase, method of, 912–920  
 stationary values  
   of functions of  
     one real variable, 50–52  
     several real variables, 162–167  
   of integrals, 776  
   under constraints, *see* Lagrange undetermined multipliers  
 statistical tests, and hypothesis testing, 1278  
 statistics, 1119, 1221–1298  
   describing data, 1222–1229  
   estimating parameters, 1229–1255, 1298  
 steepest descents, method of, 908–912  
 Stirling's  
   approximation, 637, 1185  
   asymptotic series, 637  
 Stokes constant, in Stokes phenomenon, 904  
 Stokes line, 899, 903  
 Stokes phenomenon, 903  
   dominant term, 904  
   Stokes constant, 904  
   subdominant term, 904  
 Stokes' equation, 643, 799, 888–894  
   Airy integrals, 890–894  
   qualitative solutions, 888  
   series solution, 890  
 Stokes' theorem, 388, 406–409  
   for tensors, 955  
   physical applications, 408  
   related theorems, 407  
 strain tensor, 953  
 stratified sampling, in Monte Carlo methods, 1012  
 streamlines and complex potentials, 873  
 stress tensor, 953  
 stress waves, 980  
 string  
   loaded, 798  
   plucked, 770  
   transverse vibrations of, 676, 789  
 Student's *t*-distribution  
   normalisation, 1286  
   plots, 1287  
 Student's *t*-test, 1284–1290  
   comparison of means, 1289  
   critical points table, 1288  
   one- and two-tailed confidence limits, 1288  
 Sturm–Liouville equations, 564  
   boundary conditions, 564  
   examples, 566  
     associated Laguerre, 566, 622  
     associated Legendre, 566, 590, 591  
     Bessel, 566, 608–611  
     Chebyshev, 566, 599  
     confluent hypergeometric, 566  
     Hermite, 566  
     hypergeometric, 566  
     Laguerre, 566, 619  
     Legendre, 566, 583  
   manipulation to self-adjoint form, 565–568  
   natural interval, 565, 567  
   two independent variables, 801  
   variational approach, 790–795  
   weight function, 790  
   zeros of eigenfunctions, 573  
 subdominant term, in Stokes phenomenon, 904  
 subgroups, 1061–1063  
   index, 1066  
   normal, 1063  
   order, 1061  
     Lagrange's theorem, 1065  
   proper, 1061  
   trivial, 1061  
 submatrices, 267  
 subscripts and superscripts, 928  
   contra- and covariant, 956  
   covariant derivative, 969  
   dummy, 928  
   free, 928  
   partial derivative, 969  
   summation convention, 928, 955  
 substitution, integration by, 65–67  
 summation convention, 928, 955  
 summation of series, 116–124  
   arithmetic, 117  
   arithmetico-geometric, 118  
   contour integration method, 882  
   difference method, 119  
   Fourier series method, 427  
   geometric, 117  
   powers of natural numbers, 121  
   transformation methods, 122–124  
     differentiation, 122  
     integration, 122  
     substitution, 123  
 superposition methods  
   for ODE, 554, 568–571  
   for PDE, 717–724  
 surface integrals  
   and divergence theorem, 401  
   Archimedean upthrust, 396, 410  
   of scalars, vectors, 389–396  
   physical examples, 395  
 surfaces, 345–347  
   area of, 346  
     cone, 74  
     solid, and Pappus' theorem, 195–197  
     sphere, 346  
   coordinate curves, 346  
   normal to, 346, 350  
   of revolution, 74  
   parametric equations, 345  
   quadratic, 292  
   tangent plane, 346  
 SVD, *see* singular value decomposition  
 symmetric functions, 416  
   and Fourier series, 419  
   and Fourier transforms, 445  
 symmetric matrices, 270  
   general properties, *see* Hermitian matrices

## INDEX

- symmetric tensors, 938  
 symmetry in equivalence relations, 1064  
 symmetry operations  
   on molecules, 1041  
   order of application, 1044  
  
*t*-test, *see* Student's *t*-test  
*t* substitution, 65  
 $\tan^{-1} x$ , Maclaurin series for, 140  
 tangent planes to surfaces, 346  
 tangent vectors to space curves, 342  
 tanh, hyperbolic tangent, *see* hyperbolic functions  
 Taylor series, 136–141  
   and finite differences, 1019, 1026  
   and Taylor's theorem, 136–139, 853  
   approximation errors, 139  
   in numerical methods, 992, 1003  
   as solution of ODE, 1023  
   for functions of a complex variable, 853–855  
   for functions of several real variables, 160–162  
   remainder term, 138  
   required properties, 136  
   standard forms, 136  
 tensors, *see* Cartesian tensors *and* Cartesian tensors, particular *and* general tensors  
 test statistic, 1278  
 tetrahedral group, 1115  
 tetrahedron  
   mass of, 194  
   volume of, 192  
 thermodynamics  
   first law of, 176  
   Maxwell's relations, 176–178  
 top-hat function, *see* rectangular distribution  
 torque, vector representation of, 223  
 torsion of space curves, 342  
 total derivative, 154  
 total differential, 154  
 trace of a matrix, 258  
   and second-order tensors, 939  
   as sum of eigenvalues, 280, 287  
   invariance under similarity transformations, 284, 1092  
   trace formula, 287  
 transcendental equations, 986  
 transformation matrix, 283, 289  
 transformations  
   active and passive, 948  
   conformal, 839–844  
   coordinate, *see* coordinate transformations  
   similarity, *see* similarity transformations  
 transforms, integral, *see* integral transforms *and* Fourier transforms *and* Laplace transforms  
 transients, and the diffusion equation, 723  
 transitivity in equivalence relations, 1064  
 transpose of a matrix, 250, 255  
   product rule, 256  
 transverse vibrations  
   membrane, 677, 739, 768  
   rod, 769  
   string, 676  
 trapezium rule, 1002–1004  
 trial functions  
   for eigenvalue estimation, 793  
   for particular integrals of ODE, 494  
 trials, 1119  
 triangle inequality, 246, 559  
 triangle, centroid of, 216  
 triangular matrices, 269  
 tridiagonal matrices, 998–1000, 1030, 1033  
 trigonometric identities, 10–15  
 triple integrals, *see* multiple integrals  
 triple scalar product, *see* scalar triple product  
 triple vector product, *see* vector triple product  
 turning point, 50  
  
 uncertainty principle (Heisenberg), 435–437  
   from commutator, 664  
 undetermined coefficients, method of, 494  
 undetermined multipliers, *see* Lagrange undetermined multipliers  
 uniform distribution, 1194  
 union  $\cup$ , probability for, 1121, 1125, 1128  
 uniqueness theorem  
   Klein–Gordon equation, 711  
   Laplace equation, 741  
   Poisson equation, 705–707  
 unit step function, *see* Heaviside function  
 unit vectors, 219  
 unitary  
   matrices, 271  
   eigenvalues and eigenvectors, 278  
   representations, 1086  
   transformations, 285  
 upper triangular matrices, 269  
  
 variable end-points, 782–785  
 variable, dummy, 61  
 variables, separation of, *see* separation of variables  
 variance  $\sigma^2$ , 1146  
   from MGF, 1163  
   from PGF, 1159  
   of dependent RV, 1203  
   of sample, 1224  
 variation of parameters, 508–510  
 variation, constrained, 785–787  
 variational principles, physical, 787–790  
   Fermat, 787  
   Hamilton, 788  
 variations, calculus of, *see* calculus of variations  
 vector operators, 347–369  
   acting on sums and products, 354  
   combinations of, 355–357  
   curl, 353, 368  
   del  $\nabla$ , 348  
   del squared  $\nabla^2$ , 352  
   divergence (div), 352  
   geometrical definitions, 398–400

## INDEX

- gradient operator (grad), 348–352, 367  
 identities, 356, 978  
 Laplacian, 352, 368  
 non-Cartesian, 357–369  
 tensor forms, 971–975  
   curl, 974  
   divergence, 972  
   gradient, 972  
   Laplacian, 973  
 vector product, 222–224  
   anticommutativity, 222  
   definition, 222  
   determinant form, 224  
   in Cartesian coordinates, 224  
   non-associativity, 222  
 vector spaces, 242–247, 1113  
   associativity of addition, 242  
   basis vectors, 243  
   commutativity of addition, 242  
   complex, 242  
   defining properties, 242  
   dimensionality, 243  
   group actions on, 1088  
   inequalities: Bessel, Schwarz, triangle, 246  
   invariant, 1088, 1113  
   matrices as an example, 252  
   of infinite dimensionality, 556–559  
     associativity of addition, 556  
     basis functions, 556  
     commutativity of addition, 556  
     defining properties, 556  
     Hilbert spaces, 557–559  
     inequalities: Bessel, Schwarz, triangle, 559  
   parallelogram equality, 247  
   real, 242  
   span of a set of vectors in, 242  
 vector triple product, 226  
   identities, 226  
   non-associativity, 226  
 vectors  
   as first-order tensors, 932  
   as geometrical objects, 241  
   base, 336  
   column, 250  
   compared with scalars, 212  
   component form, 217  
   examples of, 212  
   graphical representation of, 212  
   irrotational, 353  
   magnitude of, 218  
   non-Cartesian, 336, 358, 362  
   notation, 212  
   polar and axial, 949  
   solenoidal, 352, 389  
   span of, 242  
 vectors, algebra of, 212–234  
   addition and subtraction, 213  
     in component form, 218  
   angle between, 221  
   associativity of addition and subtraction, 213  
   commutativity of addition and subtraction, 213  
   multiplication by a complex scalar, 222  
   multiplication by a scalar, 214  
   multiplication of, *see* scalar product *and* vector product  
   outer product, 936  
 vectors, applications  
   centroid of a triangle, 216  
   equation of a line, 226  
   equation of a plane, 227  
   equation of a sphere, 228  
   finding distance from a  
     line to a line, 231  
     line to a plane, 332  
     point to a line, 229  
     point to a plane, 230  
   intersection of two planes, 228  
 vectors, calculus of, 334–369  
   differentiation, 334–339, 344  
   integration, 339  
   line integrals, 377–389  
   surface integrals, 389–396  
   volume integrals, 396  
 vectors, derived quantities  
   curl, 353  
   derivative, 334  
   differential, 338, 344  
   divergence (div), 352  
   reciprocal, 233, 366, 955, 959  
   vector fields, 347  
     curl, 406  
     divergence, 352  
     flux, 395  
     rate of change, 350  
 vectors, physical  
   acceleration, 335  
   angular momentum, 238  
   angular velocity, 223, 238, 353  
   area, 393–395, 408  
   area of parallelogram, 223, 224  
   force, 212, 213, 220  
   moment or torque of a force, 223  
   velocity, 335  
   velocity vectors, 335  
 Venn diagrams, 1119–1124  
 vibrations  
   internal, *see* normal modes  
   longitudinal, in a rod, 677  
   transverse  
     membrane, 677, 739, 768, 799, 801  
     rod, 769  
     string, 676, 789  
 Volterra integral equation, 804, 805  
   differentiation methods, 812  
   Laplace transform methods, 810  
 volume elements  
   curvilinear coordinates, 365  
   cylindrical polars, 360  
   spherical polars, 205, 362

## INDEX

- volume integrals, 396  
   and divergence theorem, 401  
 volume of  
   cone, 75  
   ellipsoid, 207  
   parallelepiped, 225  
   rhomboid, 237  
   tetrahedron, 192  
 volumes  
   as surface integrals, 397, 401  
   in many dimensions, 210  
   of regions, using multiple integrals, 191–193  
 volumes of revolution, 75  
   and surface area & centroid, 195–197
- wave equation, 676, 688, 790  
   boundary conditions, 693–695  
   characteristics, 704  
   from Maxwell's equations, 373  
   in one dimension, 689, 693–695  
   in three dimensions, 695, 714, 737  
   standing waves, 693  
 wave number, 437, 693n  
 wave packet, 436  
 wave vector,  $\mathbf{k}$ , 437  
 wavefunction of electron in hydrogen atom, 208  
 Weber functions  $Y_\nu(x)$ , 607  
 wedge product, *see* vector product  
 weight  
   of relative tensor, 964  
   of variable, 477  
 weight function, 555, 790  
 Wiener–Kinchin theorem, 450  
 WKB methods, 895–905  
   accuracy, 902  
   general solutions, 897  
   phase memory, 895  
   the Stokes phenomenon, 903  
 work done  
   by force, 381  
   vector representation, 220  
 Wronskian  
   and Green's functions, 527  
   for second solution of ODE, 544, 580  
   from ODE, 532  
   test for linear independence, 491, 532
- X-ray scattering, 237
- $Y^m(\theta, \phi)$ , *see* spherical harmonics  
 $Y_\nu(x)$ , Bessel functions of second kind, 607  
 Young's modulus, 677, 953
- $z$ , as a complex number, 84  
 $z^*$ , as complex conjugate, 89–91  
 zero (null)  
   matrix, 254, 255  
   operator, 249  
   unphysical state  $|\emptyset\rangle$ , 650  
   vector, 214, 242, 556  
   zero-order tensors, 932–935  
   zeros of a function of a complex variable, 839  
     location of, 879–882, 921  
     order, 839, 856  
     principle of the argument, 880  
     Rouché's theorem, 880, 882  
   zeros of Sturm-Liouville eigenfunctions, 573  
   zeros, of a polynomial, 2  
   zeta series (Riemann), 128, 129  
    $z$ -plane, *see* Argand diagram