

PROFESSOR STEWART'S INCREDIBLE NUMBERS

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Preface

I've always been fascinated by numbers. My mother taught me to read and to count, long before I first went to school. Apparently, when I did, I came back at the end of day one complaining that 'we didn't *learn* anything!'. I suspect that my parents had been preparing me for this difficult day by telling me that I would learn all sorts of interesting things, and I'd taken the propaganda a little too much to heart. But soon I was learning about planets and dinosaurs and how to make a plaster animal. And more about numbers.

I'm still enchanted by numbers, and still learning more about them. Now, I'm always quick to point out that mathematics is about many different ideas, not just numbers; for example, it's also about shapes, patterns, and probabilities—but numbers underpin the entire subject. And every number is a unique individual. A few special numbers stand out above the rest and seem to play a central role in many different areas of mathematics. The most familiar of these is π (pi), which we first encounter in connection with circles, but it has a remarkable tendency to pop up in problems that seem not to involve circles at all.

Most numbers cannot aspire to such heights of importance, but you can usually find some unusual feature of even the humblest number. In *The Hitchhiker's Guide to the Galaxy*, the number 42 was 'the answer to the great question of life, the universe, and everything.' Douglas Adams said he chose that number because a quick survey of his friends suggested it was totally boring. Actually, it's not, as the final chapter demonstrates.

The book is organised in terms of the numbers themselves, although not always in numerical order. As well as chapters 1, 2, 3, and so on, there is also a chapter 0, a chapter 42, a chapter -1 , a chapter $\frac{22}{7}$, a chapter π , a chapter 43,252,003,274,489,856,000, and a chapter $\sqrt{2}$. Clearly a lot of potential chapters never made it off the number line. Each chapter starts with a short summary of the main topics to be included. Don't worry if the summary occasionally seems cryptic, or if

it makes flat statements unsupported by any evidence: all will be revealed as you read on.

The structure is straightforward: each chapter focuses on an interesting number and explains *why* it's interesting. For instance, 2 is interesting because the odd/even distinction shows up all over mathematics and science; 43,252,003,274,489,856,000 is interesting because it's the number of ways to rearrange a Rubik cube.

Since 42 is included, it must be interesting. Well, a *bit*, anyway.

At this point I must mention Arlo Guthrie's *Alice's Restaurant Massacre*, a musical shaggy dog story that relates in great and repetitious detail events involving the dumping of garbage. Ten minutes into the song, Guthrie stops and says: 'But that's not what I came here to talk to you about.' Eventually you find out that actually it is what he came to talk about, but that garbage is only part of a greater story. Time for my Arlo Guthrie moment: this *isn't* really a book about numbers.

The numbers are the entry point, a route through which we can dive into the amazing mathematics associated with them. *Every number is special*. When you come to appreciate them as individuals, they're like old friends. Each has its own story to tell. Often that story leads to lots of other numbers, but what really matters is the mathematics that links them. The numbers are the characters in a drama, and the most important thing is the drama itself. But you can't have a drama without characters.

To avoid getting too disorganised, I've divided the book into sections according to the kind of number: small whole numbers, fractions, real numbers, complex numbers, infinity... With a few unavoidable exceptions, the material is developed in logical order, so that earlier chapters lay the groundwork for later ones, even when the topic changes completely. This requirement influences how the numbers are arranged, and it requires a few compromises. The most significant involves complex numbers. They appear very early, because I need them to discuss some features of more familiar numbers. Similarly, an advanced topic occasionally crops up somewhere because that's the only sensible place to mention it. If you meet one of these passages and find it hard going, skip it and move on. You can come back to it later.

This book is a companion volume to my iPad app *Incredible*

Numbers. You don't need the app to read the book, and you don't need the book to use the app. In fact, the overlap between them is quite small. Each complements the other, because each medium can do things that the other can't.

Numbers truly are incredible—not in the sense that you don't believe anything you hear about them, but in the positive sense: they have a definite wow factor. And you can experience it without doing any sums. You can see how numbers evolved historically, appreciate the beauty of their patterns, find out how they are used, marvel at the surprises: 'I never knew 56 was so fascinating!' But it is. It really is.

So are all the others. Including 42.

Numbers

1, 2, 3, 4, 5, 6, 7, ... What could be simpler than that? Yet it is numbers, perhaps more than anything else, that have enabled humanity to drag itself out of the mud and aim at the stars.

Individual numbers have their own characteristic features, and lead to a variety of areas of mathematics. Before examining them one by one, however, it's worth a quick look at three big questions. How did numbers originate? How did the number concept develop? And what *are* numbers?

The Origin of Numbers

About 35,000 years ago, in the Upper Palaeolithic, an unknown human carved 29 marks in the fibula (calf bone) of a baboon. It was found in a cave in the Lebombo Mountains of Swaziland, and is called the Lebombo bone. It is thought to be a tally stick: something that records numbers as a series of notches: |, ||, |||, and so on. There are 29.5 days in a lunar month, so it could be a primitive lunar calendar—or a record of the female menstrual cycle. Or a random collection of cut-marks, for that matter. A bone doodle.

The wolf bone, another tally stick with 55 marks, was found in Czechoslovakia in 1937 by Karl Absolon. It is about 30,000 years old.

In 1960 the Belgian geologist Jean de Heinzelin de Braucourt discovered a notched baboon fibula among the remains of a tiny fishing community that had been buried by an erupting volcano. The location was what is now Ishango, on the border between Uganda and Congo. The bone has been dated to about 20,000 years ago.

The simplest interpretation of the Ishango bone is again a tally stick. Some anthropologists go further and detect elements of arithmetical structure, such as multiplication, division, and prime numbers; some think it is a six-month lunar calendar; some are convinced that the marks were made to provide a good grip on a bone tool, and have no mathematical significance.



Fig 1 Front and rear of Ishango bone in the Museum of Natural Sciences, Brussels.

It's certainly intriguing. There are three series of notches. The central series uses the numbers 3, 6, 4, 8, 10, 5, 7. Twice 3 is 6, twice 4 is 8, and twice 5 is 10; however, the order for the final pair is the other way round, and 7 doesn't fit the pattern at all. The left-hand series is 11, 13, 17, 19: the prime numbers from 10 to 20. The right-hand series supplies the odd numbers 11, 21, 19, 9. The left- and right-hand series each add to 60.

One problem with the interpretation of patterns like this is that it's difficult not to find a pattern in *any* series of smallish numbers. For instance, Table 1 shows a list of the areas of ten islands in the Bahamas, namely numbers 11–20 in terms of total area. To jumble up the list I've put them in alphabetical order. I promise you: this was the first thing I tried. Admittedly, I'd have replaced it by something else if it hadn't made my point—but it did, so I didn't.

What do we notice in this 'pattern' of numbers? There are lots of short sequences with common features:

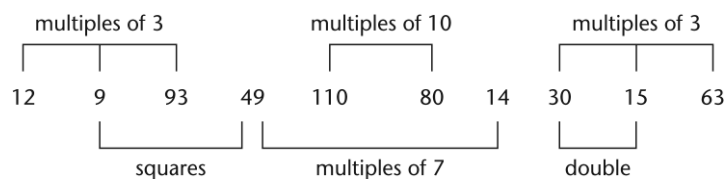


Fig 2 Some apparent patterns in the areas of Bahama islands.

For a start, there's a beautiful symmetry to the whole list. At each end there's a triple of multiples of 3. In the middle, there's a pair of

Name	area in square miles
Berry	12
Bimini	9
Crooked Island	93
Little Inagua	49
Mayaguana	110
New Providence	80
Ragged Island	14
Rum Cay	30
Samana Cay	15
San Salvador Island	63

Table 1

multiples of 10, separating two multiples of 7. Moreover, two squares, $9 = 3^2$ and $49 = 7^2$, occur—both squares of *primes*. Another adjacent pair consists of 15 and 30, one twice the other. In the sequence 9–93–49, all numbers have a digit 9. The numbers become alternately larger and smaller, except for 110–80–14. Oh, and did you notice that *none* of these ten numbers is prime?

Enough said. Another problem with the Ishango bone is the virtual impossibility of finding extra evidence to support any specific interpretation. But the markings on it are certainly intriguing. Number puzzles always are. So here's something less contentious.

Ten thousand years ago people in the Near East were using clay tokens to record numbers, perhaps for tax purposes or as proof of ownership. The oldest examples are from Tepe Asiab and Ganj-i-Dareh Tepe, two sites in the Zagros Mountains of Iran. The tokens were small lumps of clay of various shapes, some bearing symbolic marks. A ball marked + represented a sheep; seven such balls recorded seven sheep. To avoid making vast numbers of tokens, a different type of token stood for ten sheep. Yet another represented ten goats, and so on. The archaeologist Denise Schmandt-Besserat deduced that the tokens represented basic staples of the time: grain, animals, jars of oil.

By 4000 BC the tokens were being strung on a string like a necklace. However, it was easy to change the numbers by adding or removing tokens, so a security measure was introduced. The tokens

were wrapped in clay, which was then baked. A dispute about numbers could be resolved by breaking open the clay envelope. From 3500 BC, to avoid unnecessary breakage, the bureaucrats of ancient Mesopotamia inscribed symbols on the envelope, listing the tokens inside it.

Then one bright spark realised that the symbols made the tokens redundant. The upshot was a system of written number symbols, paving the way for all subsequent systems of number notation, and possibly of writing itself.



Fig 3 Clay envelope and accountancy tokens, Uruk period, from Susa.

This isn't primarily a history book, so I'll look at later notational systems as they arise in connection with specific numbers. For instance, ancient and modern decimal notations are tackled in chapter [10]. However, as the great mathematician Carl Friedrich Gauss once remarked, the important thing is not notations, but notions. Subsequent topics will make more sense if they are viewed within the context of humanity's changing conception of numbers. So we'll start with a quick trip through the main number systems and some important terminology.

The Ever-Growing Number System

We tend to think of numbers as something fixed and immutable: a feature of the natural world. Actually, they are human inventions—but very useful ones, because they represent important aspects of nature. Such as how many sheep you own, or the age of the universe. Nature repeatedly surprises us by opening up new questions, whose answers sometimes require new mathematical concepts. Sometimes the internal demands of mathematics hint at new, potentially useful structures. From time to time these hints and problems have led mathematicians to extend the number system by inventing new kinds of numbers.

We've seen how numbers first arose as a method for counting things. In early ancient Greece, the list of numbers started 2, 3, 4, and so on: 1 was special, not 'really' a number. Later, when this convention started to look really silly, 1 was deemed to be a number as well.

The next big step forward in the enlargement of the number system was to introduce fractions. These are useful if you want to divide some commodity among several people. If three people get equal shares of two bushels of grain, each receives $\frac{2}{3}$ of a bushel.



Fig 4 Left: Egyptian hieroglyphs for $\frac{2}{3}$ and $\frac{3}{4}$. Middle: Wadjet eye. Right: Fraction hieroglyphs derived from it.

The ancient Egyptians represented fractions in three different ways. They had special hieroglyphs for $\frac{2}{3}$ and $\frac{3}{4}$. They used various portions of the eye of Horus, or wadjet eye, to represent 1 divided by the first six powers of 2. Finally, they devised symbols for unit fractions, those of the form 'one over something': $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and so on. They expressed all

other fractions as sums of distinct unit fractions. For instance,

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$$

It's not clear why they didn't write $\frac{2}{3}$ as $\frac{1}{3} + \frac{1}{3}$, but they didn't.

The number zero came much later, probably because there was little need for it. If you don't have any sheep, there's no need to count them or list them. Zero was first introduced as a symbol, and was not thought to be a number as such. But when [see -1] Chinese and Hindu mathematicians introduced negative numbers, 0 had to be considered a number as well. For example, $1 + (-1) = 0$, and the sum of two numbers must surely count as a number.

Mathematicians call the system of numbers

$$0, 1, 2, 3, 4, 5, 6, 7, \dots$$

the *natural numbers*, and when negative numbers are included we get the *integers*

$$\dots, -3, -2, -1, 0, 1, 2, 3, \dots$$

Fractions, zero, and negative fractions form the *rational numbers*.

A number is *positive* if it is bigger than zero, and *negative* if it is smaller than zero. So every number (be it an integer or rational) falls into exactly one of three distinct categories: positive, negative, or zero. The counting numbers

$$1, 2, 3, 4, 5, 6, 7, \dots$$

are the positive integers. This convention leads to one rather clumsy piece of terminology: the natural numbers or whole numbers

$$0, 1, 2, 3, 4, 5, 6, 7, \dots$$

are often referred to as the *non-negative* integers. Sorry about that.

For a long time, fractions were as far as the number concept went. But the ancient Greeks proved that the square of a fraction can never be exactly equal to 2. Later this was expressed as 'the number $\sqrt{2}$ is irrational', that is, not rational. The Greeks had a more cumbersome way of saying that, but they knew that $\sqrt{2}$ must exist: by Pythagoras's theorem, it is the length of the diagonal of a square of side 1. So more numbers are needed: rationals alone can't hack it. The Greeks found a

complicated geometric method for dealing with irrational numbers, but it wasn't totally satisfactory.

The next step towards the modern concept of number was made possible by the invention of the decimal point (·) and decimal notation. This made it possible to represent irrational numbers to very high accuracy. For example,

$$\sqrt{2} \sim 1.4142135623$$

correct to 10 decimal places (here and elsewhere the symbol \sim means 'is approximately equal to'). This expression is not exact: its square is actually

$$1.9999999979325598129$$

A better approximation, correct to 20 decimal places, is

$$\sqrt{2} \sim 1.41421356237309504880$$

but again this is not exact. However, there is a rigorous logical sense in which an infinitely long decimal expansion *is* exact. Of course such expressions can't be written down in full, but it's possible to set up the ideas so that they make sense.

Infinitely long decimals (including ones that stop, which can be thought of as decimals ending in infinitely many 0s) are called *real numbers*, in part because they correspond directly to measurements of the natural world, such as lengths or weights. The more accurate the measurement, the more decimal places you need; to get an exact value, you need infinitely many. It is perhaps ironic that 'real' is defined by an infinite symbol which can't actually be written down in full. Negative real numbers are also allowed.

Until the eighteenth century no other mathematical concepts were considered to be genuine numbers. But even by the fifteenth century, a few mathematicians were wondering whether there might be a new kind of number: the square root of minus one. That is, a number that gives -1 when you multiply it by itself. At first sight this is a crazy idea, because the square of any real number is positive or zero. However, it turned out to be a good idea to press ahead regardless and equip -1 with a square root, for which Leonhard Euler introduced the symbol i . This is the initial letter of 'imaginary' (in English, Latin,